A slide show of our Lecture Note

Fuzzy Data Processing

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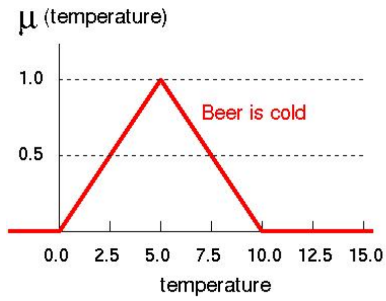
I. Fuzzy Basic Arithmetics

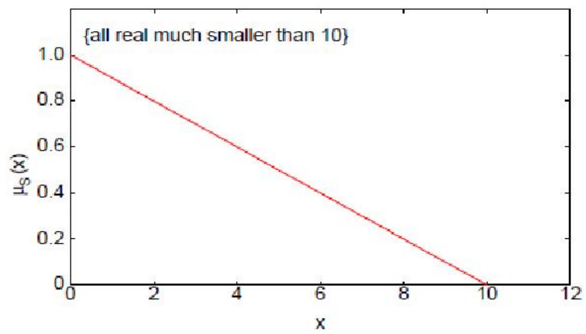
Membership Function

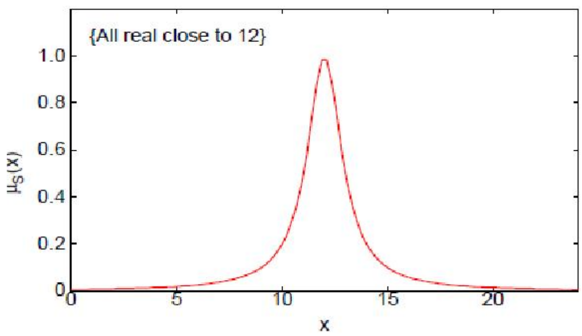
In Fuzzy logic the probability of "*how likely A is true*" is called membership value of A and expressed as μ_A. E.g., assuming A = "beer is cold," $\mu_A = 1$ when temperature of beer is 5°C, while $\mu_A = 0.5$ when temperature of beer is 10°C, and $\mu_A = 0$ when temperature of beer is 15°C.



Other types of Membership Function

Is This beer cold?





AND and OR

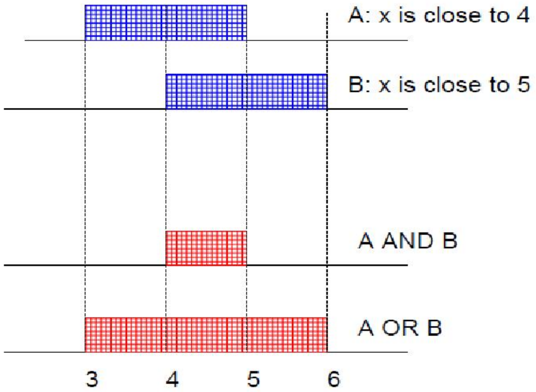
Membership of A AND B and A OR B are given, respectively, as

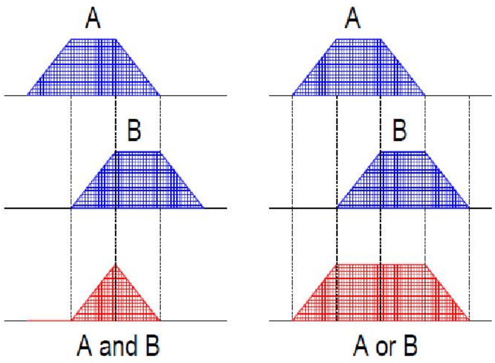
$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

and

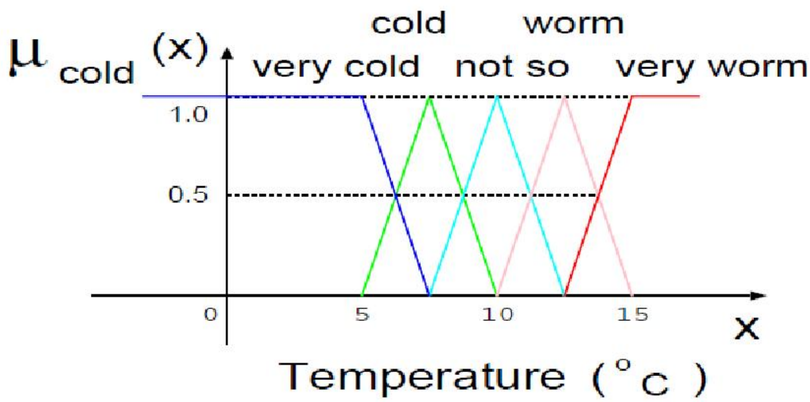
$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

AND and OR - Crisp/Fuzzy

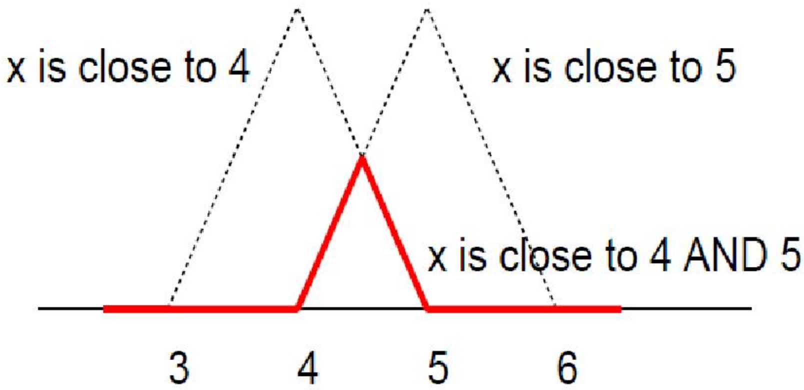




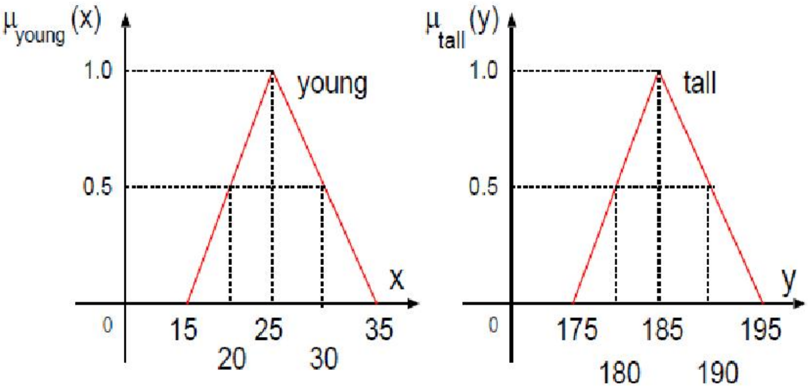
Temperature of Beer



Membership of AND & OR

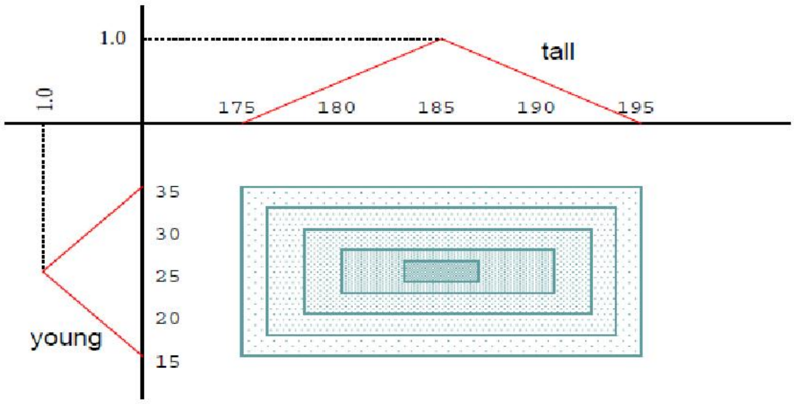


Young & Tall



$\mu_{young}(x)$
1.0
0.5
young
x
0
15
20
25
30
35
$\mu_{tall}(y)$
1.0
0.5
tall
y
0
175
180
185
190
195

A Representation of Membership of Young AND Tall



IF-THEN

Membership of IF A THEN B has proposed by many but here we use this Larsen's proposal.

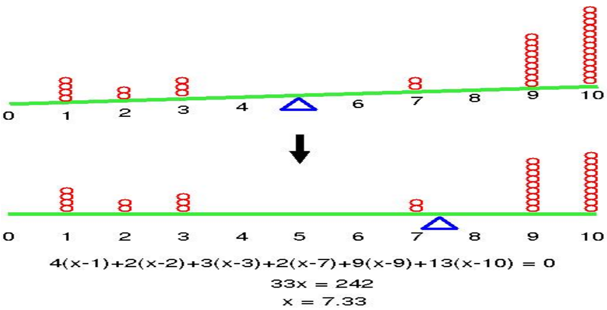
$$\mu_{A\to B}(x) = \mu_A(x) \times \mu_B(x)$$

De-fuzzification

When A has some different possibility, we determine most possible value of A by calculating the center of gravity of these membership values.

$$\sum_i \mu_{A_i} \times (x - x_i) = 0$$

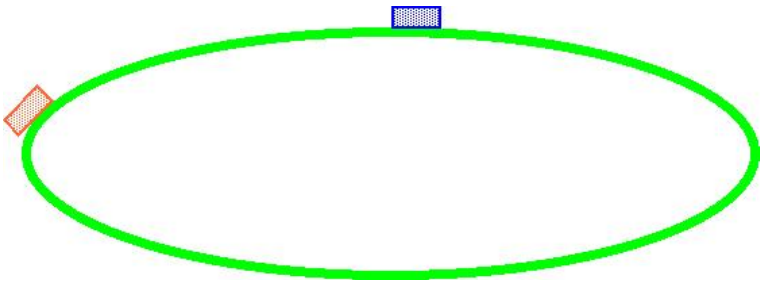
E.g.



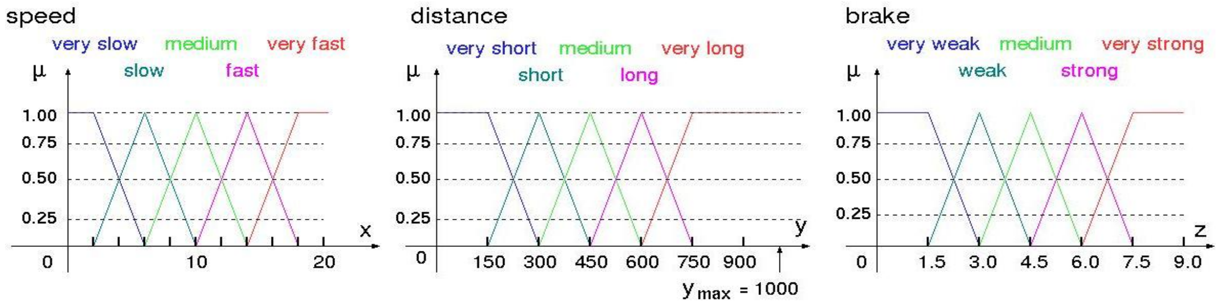
II. Fuzzy Controller

Controll two metro cars

Let's create a virtual metro system with 2 cars on a loop line with 1000 pixels. Each car has a pair of 3 parameters of speed x, distance to the car in front y and strength of brake z.

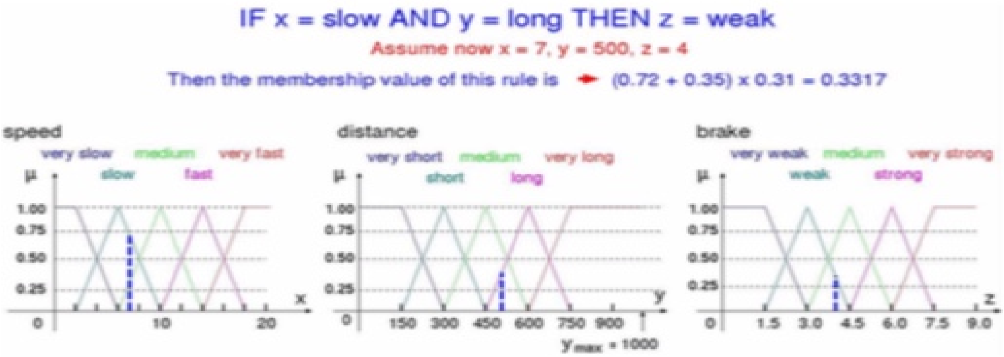


Membership function of Speed, Distance and Brake assumed here.



Membership value of a rule with specific speed, distance and brake.

E.g.
The membership value below implies how this brake = 4 will be likely
when speed = 7 and distance = 500 under the rule below.



An example of Membership value of one rule

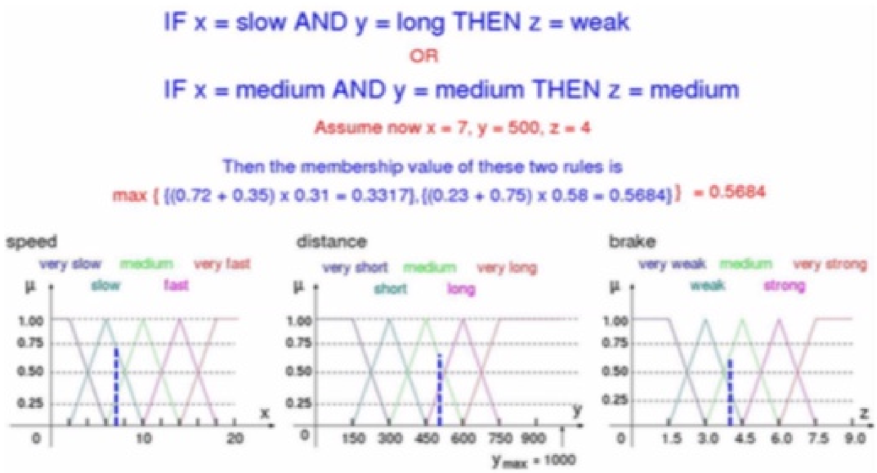
Membership value of brake = 0,1,2,3,4,5,6,7,8,9 when speed = 20 and distance = 650 under the rule
IF speed = medium AND distance = long THEN brake = medium.

Rule: If speed is medium, distance is long, then break is

Speed	Mui	Distance	Mui	Break	Mui	TotalM
0	0	0	0	0	0	0
-	-	-	-	-	-	-
-	-	-	-	-	-	-
-	-	-	-	-	-	-
20	0,5	650	0,5	0	0	0
20	0,5	650	0,5	1	0	0
20	0,5	650	0,5	2	0	0
20	0,5	650	0,5	3	0	0
20	0,5	650	0,5	4	0,5	0,25
20	0,5	650	0,5	5	1	0,5
20	0,5	650	0,5	6	0,5	0,25
20	0,5	650	0,5	7	0	0
20	0,5	650	0,5	8	0	0
20	0,5	650	0,5	9	0	0
-	-	-	-	-	-	-
-	-	-	-	-	-	-
-	-	-	-	-	-	-
40	0	1000	0	9	0	0

From the work by Korol Andrey (2015 Fall)

Membership value of two rules



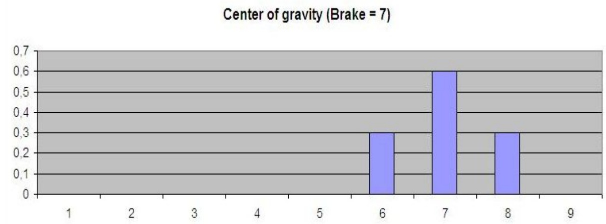
Membership value of 3 rules for a pair of speed & distance

Speed	Distance	Brake	Rule 1: IF x=medium AND y=small THEN z=strong				Rule 2: IF x=medium AND y=medium THEN z=medium				Rule 3: IF x=medium AND y=large THEN z=week				Max of rules
			mSp1	mDs1	mBr1	min(mSp,mDs)*mBr	mSp2	mDs2	mBr2	min(mSp,mDs)*mBr	mSp3	mDs3	mBr3	min(mSp,mDs)*mBr	
		0	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		1	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		2	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0,25	0,1875	0,1875
		3	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	1	0,75	0,75
		4	0,75	0	0	0	0,75	0,25	0,75	0,1875	0,75	0,75	0,25	0,1875	0,1875
11,00	550,00	5	0,75	0	0,3	0	0,75	0,25	0,75	0,1875	0,75	0,75	0	0	0,1875
		6	0,75	0	1	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		7	0,75	0	0,3	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		8	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		9	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		10	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0

From the work by Yulia Bogutskaya (2016 Fall)

Defuzzified value of break for a pair of a speed and a distance

			Speed is very slow AND Distance is very short THEN Brake is strong			Speed is very slow AND Distance is short THEN Brake is strong			Speed is medium AND Distance is short THEN Brake is very strong			
Speed	Distance	Brake	μ1 Speed	μ1 Distance	μ1 Brake	μ2 Speed	μ2 Distance	μ3 Brake	μ3 Speed	μ3 Distance	μ3 Brake	Result
0	150	0	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	1	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	2	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	3	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	4	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	5	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	6	1	0.4	0.5	1	0.6	0.5	0	0.6	0	0.3
0	150	7	1	0.4	1	1	0.6	1	0	0.6	0	0.6
0	150	8	1	0.4	0.5	1	0.6	0.5	0	0.6	0.5	0.3
0	150	9	1	0.4	0	1	0.6	0	0	0.6	1	0



From the work by Kuchur Alexander (2015 Fall)

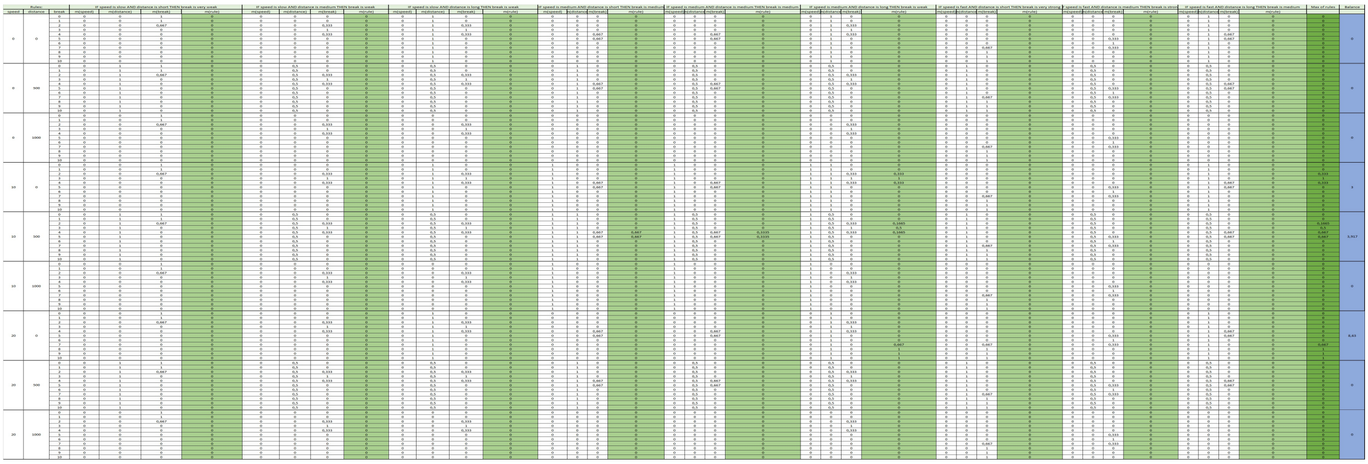
Membership value of 3 rules for 3 pairs of speed & distance

Speed	Distance	Brake	Rule 1: IF x=medium AND y=small THEN z=strong				Rule 2: IF x=medium AND y=medium THEN z=medium				Rule 3: IF x=medium AND y=large THEN z=week				Max of rules	Balance
			mSp1	mDs1	mBr1	min(mSp;mDs)*mBr	mSp2	mDs2	mBr2	min(mSp;mDs)*mBr	mSp3	mDs3	mBr3	min(mSp;mDs)*mBr		
		0	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		1	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		2	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0,25	0,125	0,125	
		3	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	1	0,5	0,5	
		4	0,75	0	0	0	0,75	0,5	0,75	0,375	0,75	0,5	0,25	0,125	0,375	
11,00	500,00	5	0,75	0	0,3	0	0,75	0,5	0,75	0,375	0,75	0,5	0	0	0,375	3,727273
		6	0,75	0	1	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		7	0,75	0	0,3	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		8	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		9	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		10	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		0	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		1	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		2	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0,25	0,1875	0,1875	
		3	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	1	0,75	0,75	
		4	0,75	0	0	0	0,75	0,25	0,75	0,1875	0,75	0,75	0,25	0,1875	0,1875	
11,00	550,00	5	0,75	0	0,3	0	0,75	0,25	0,75	0,1875	0,75	0,75	0	0	0,1875	3,285714
		6	0,75	0	1	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		7	0,75	0	0,3	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		8	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		9	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		10	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		0	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		1	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		2	0,75	0	0	0	0,75	0	0	0	0,75	1	0,25	0,1875	0,1875	
		3	0,75	0	0	0	0,75	0	0	0	0,75	1	1	0,75	0,75	
		4	0,75	0	0	0	0,75	0	0,75	0	0,75	1	0,25	0,1875	0,1875	
11,00	600,00	5	0,75	0	0,3	0	0,75	0	0,75	0	0,75	1	0	0	0	3
		6	0,75	0	1	0	0,75	0	0	0	0,75	1	0	0	0	
		7	0,75	0	0,3	0	0,75	0	0	0	0,75	1	0	0	0	
		8	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		9	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		10	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	

From the work by Yulia Bogutskaya (2016 Fall)

Membership function of 25 rules

Too small to be visible but all combination of speed, distance and brake.



From the work by Lishko Aleksandr (2016 Fall)

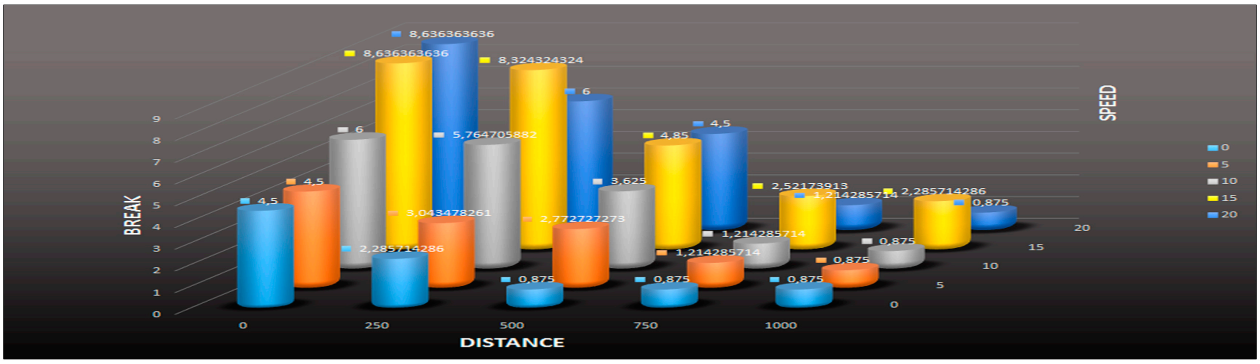
6. 3-D bar-graph of speed-distance-brake with 25 rules

speed	0	0	0	0	0	5	5	5	5	5	10	10	10
distance	0	250	500	750	1000	0	250	500	750	1000	0	250	500
break	4,5	2,285714	0,875	0,875	0,875	4,5	3,043478	2,772727	1,214286	0,875	6	5,764706	3,625

speed	10	10	15	15	15	15	15	20	20	20	20	20
distance	750	1000	0	250	500	750	1000	0	250	500	750	1000
break	1,214286	0,875	8,636364	8,324324	4,85	2,521739	2,285714	8,636364	6	4,5	1,214286	0,875

IF speed IS	vSLOW	vSLOW	vSLOW	vSLOW	vSLOW	SLOW	SLOW	SLOW	SLOW	SLOW	MEDIUM	MEDIUM	MEDIUM
AND distance IS	vSHORT	SHORT	MEDIUM	LONG	vLONG	vSHORT	SHORT	MEDIUM	LONG	vLONG	vSHORT	SHORT	MEDIUM
THEN break IS	MEDIUM	MEDIUM	WEAK	WEAK	WEAK	MEDIUM	MEDIUM	MEDIUM	WEAK	vWEAK	STRONG	STRONG	MEDIUM

IF speed IS	MEDIUM	MEDIUM	FAST	FAST	FAST	FAST	FAST	vFAST	vFAST	vFAST	vFAST	vFAST
AND distance IS	LONG	vLONG	vSHORT	SHORT	MEDIUM	LONG	vLONG	vSHORT	SHORT	MEDIUM	LONG	vLONG
THEN break IS	WEAK	vWEAK	vSTRONG	vSTRONG	STRONG	MEDIUM	WEAK	vSTRONG	vSTRONG	STRONG	MEDIUM	WEAK

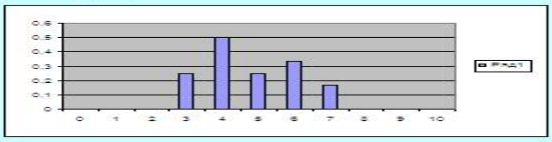


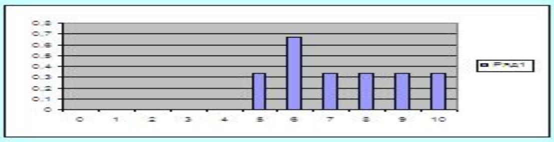
From the work by Bokhanov Evgenii (2015 Fall)

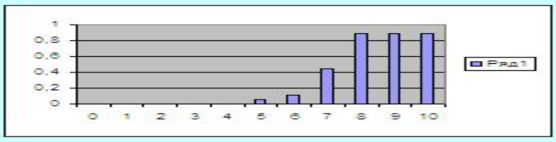
3-D surface of speed-distance-brake with limited domain

An example of how to draw for a fixed speed and three diferent value of distances

Distance	Speed	Brake	Max	Center of gravity	Final brake
400	12	0	0		4,8
400	12	1	0		4,8
400	12	2	0		4,8
400	12	3	0,25		4,8
400	12	4	0,5		4,8
400	12	5	0,25		4,8
400	12	6	0,333333		4,8
400	12	7	0,166667		4,8
400	12	8	0		4,8
400	12	9	0		4,8
400	12	10	0		4,8
400	24	0	0		7,3
400	24	1	0		7,3
400	24	2	0		7,3
400	24	3	0		7,3
400	24	4	0		7,3
400	24	5	0,333333		7,3
400	24	6	0,666667		7,3
400	24	7	0,333333		7,3
400	24	8	0,333333		7,3
400	24	9	0,333333		7,3
400	24	10	0,333333		7,3
400	36	0	0		8,6
400	36	1	0		8,6
400	36	2	0		8,6
400	36	3	0		8,6
400	36	4	0		8,6
400	36	5	0,0555556		8,6
400	36	6	0,111111		8,6
400	36	7	0,444444		8,6
400	36	8	0,888889		8,6
400	36	9	0,888889		8,6
400	36	10	0,888889		8,6



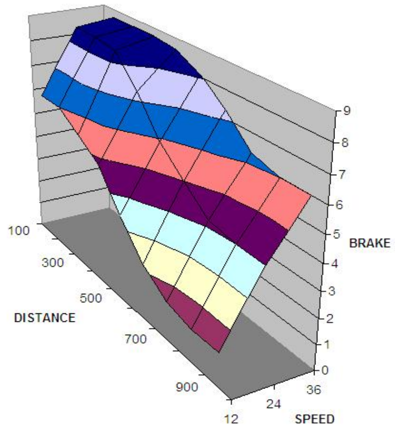




From the work by Bokhanov Evgenii (2015 Fall)

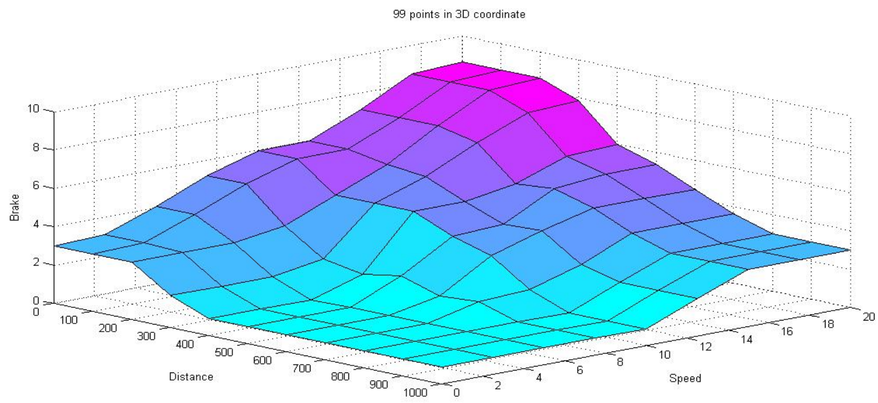
3-D surface of speed-distance-brake with limited domain (continued)

Distance	Speed	Final Brake
100	12	6
100	24	8.5
100	36	8.7
200	12	6
200	24	8.5
200	36	8.7
300	12	5.4
300	24	8.4
300	36	8.7
400	12	4.8
400	24	7.3
400	36	8.6
500	12	3.4
500	24	5.9
500	36	8.1
600	12	2.2
600	24	5
600	36	7.3
700	12	1.5
700	24	4.2
700	36	6.3
800	12	1.3
800	24	4
800	36	6
900	12	1.3
900	24	4
900	36	6



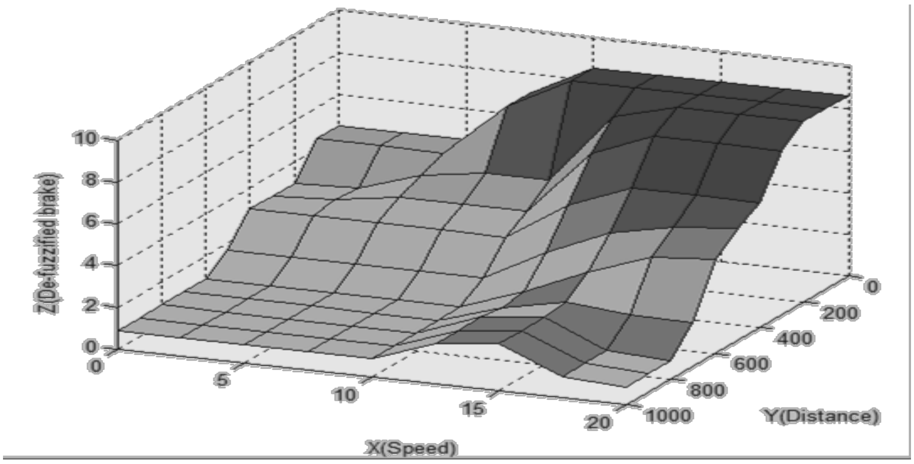
From the work by Bokhanov Evgenii (2015 Fall)

A 3-D surface of speed-distance-brake over whole domain



From the work by Yulia Bogutskaya (2016 Fall)

Another 3-D surface of speed-distance-brake over whole domain



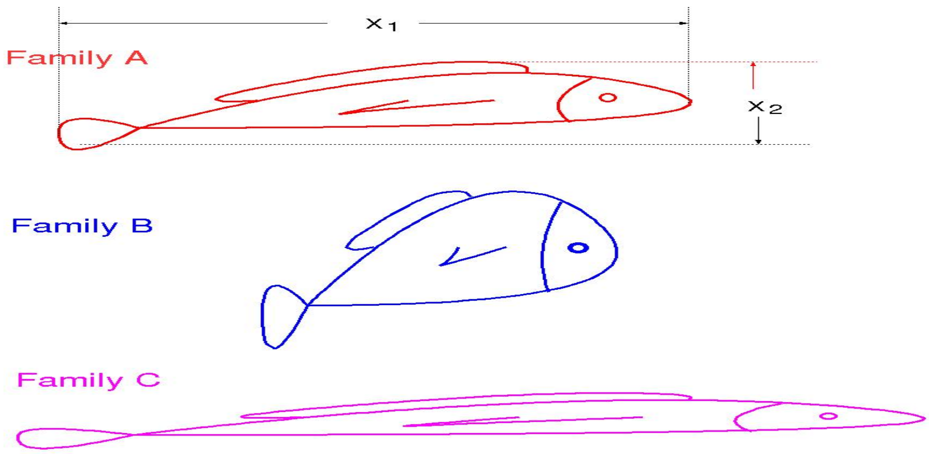
From the work by Kolesnikov Dmitry (2016 Fall)

7. Control metros by 3-D surface of speed-distance-brake

From the work by Muzyka Aleksandr (2016 Fall)

III. Fuzzy Classification

An example of classification - 3 families of fish



Rules to classify as an example

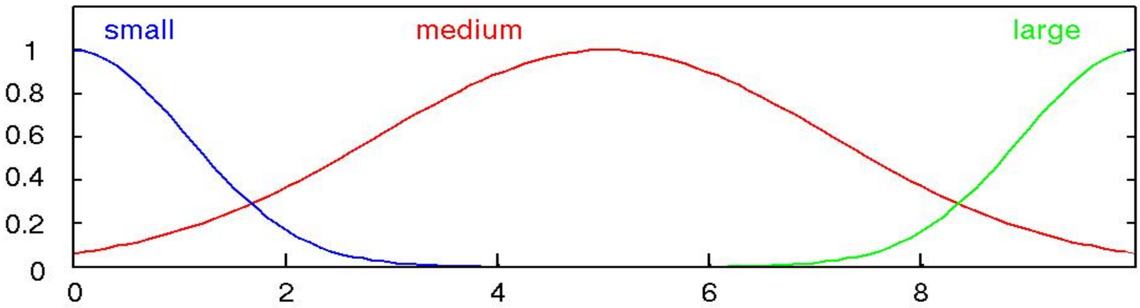
R_1: IF X_1 = medium AND X_2 = small THEN A

R_2: IF X_1 = small AND X_2 = medium THEN B

R_3: IF X_1 = large AND X_2 = small THEN C

Memership function for the size of two parts

$$\mu(x) = \exp\{-\frac{(x-(avg)^2)}{(std)^2}\}$$



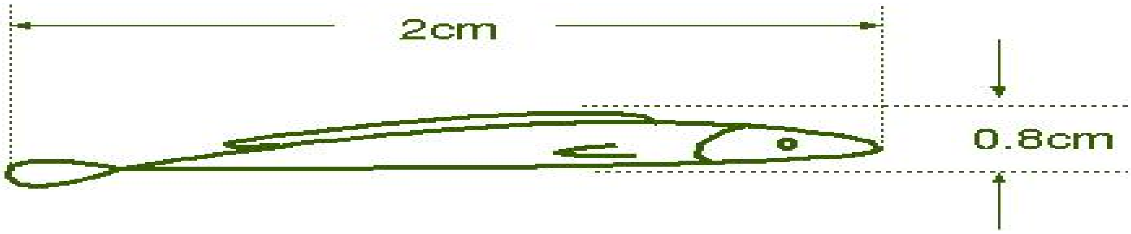
How to estimate avg and std from dataset

How we specify avg and std for each of membership function from dataset given?

Algorithm 1

1. *Select maximum data + minimum data + other randomly chose $5N - 2$ data.*
2. *Sort these $5N$ data from small to large in each attribute.*
3. *Devide the data in each attribute into 5 groups, that is, very small, small, medium, large, and very large.*
4. *Calculate average and stndard deviation in eact devision.*

Question: Which family is this new fish?



Takagi Sugeno Formula

R_k: If x_1 is A_1^k, and x_2 is A_2^k and $\cdots$ and x_N is A_N^k then y is g^k.

Takagi-Sugeno rules: Estimation of a single input

Estimation of y for an input $\mathbf{x} = (x_1, x_2, \cdots, x_N)$

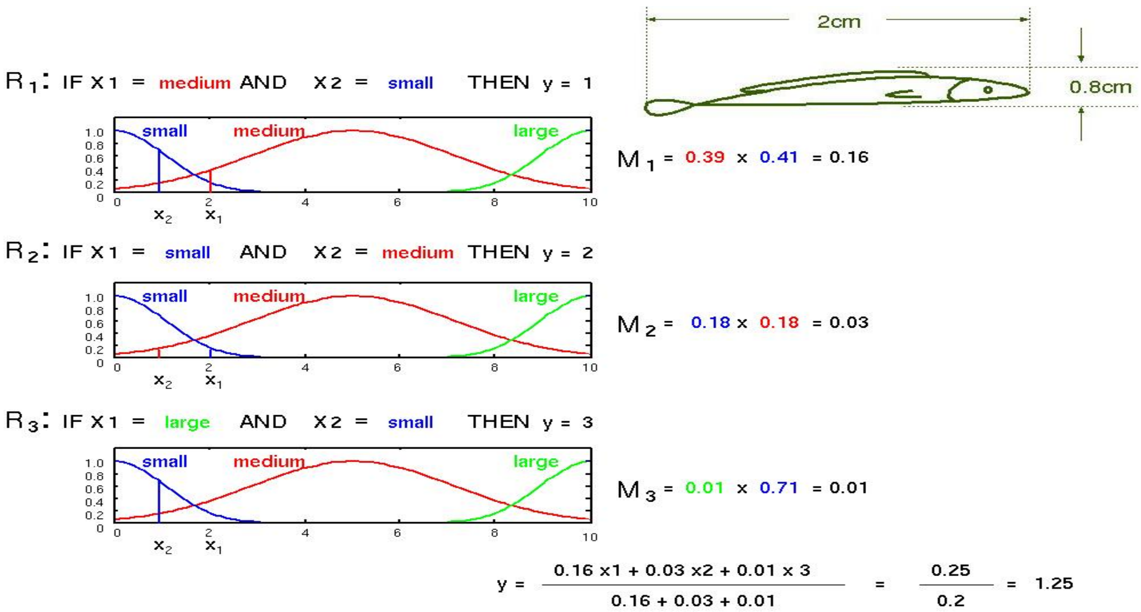
$$y_j = \frac{\sum_{k=1}^{H} (M_k(\mathbf{x}) \cdot g_k)}{\sum_{k=1}^{H} M_k(\mathbf{x})}$$

where

$$M_k(\mathbf{x}) = \prod_{i=1}^{N} \mu_{ik}(x_i)$$

where μ_{ik} is i-th attribute of k-th rule

Three rules to classify

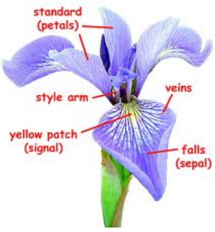


A benchmark – Iris database

Iris flower dataset (taken from University of California Urvine Machine Learning Repository) consists of three species of iris flower

setosa, versicolor and *virginica.*

Each sample represents four attributes of the iris flower
sepal-length, sepal-width, petal-length, and *petal-width.*



Iris Flower Database to design

Setosa				Versicolor				Virginica			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.56	0.66	0.20	0.08	0.84	0.66	0.67	0.52	0.85	0.57	0.84	0.72
0.62	0.70	0.22	0.04	0.66	0.61	0.57	0.56	0.91	0.82	0.88	1.00
0.68	0.84	0.22	0.08	0.63	0.45	0.51	0.40	0.82	0.73	0.74	0.80
0.61	0.77	0.23	0.08	0.75	0.68	0.61	0.60	0.81	0.61	0.77	0.76
0.61	0.68	0.20	0.04	0.76	0.50	0.58	0.40	0.86	0.68	0.80	0.84
0.54	0.68	0.16	0.04	0.77	0.66	0.68	0.56	0.72	0.57	0.72	0.80
0.73	0.91	0.17	0.08	0.71	0.66	0.52	0.52	0.73	0.64	0.74	0.96
0.72	1.00	0.22	0.16	0.85	0.70	0.64	0.56	0.81	0.73	0.77	0.92
0.68	0.89	0.19	0.16	0.71	0.68	0.65	0.60	0.82	0.68	0.80	0.72
0.65	0.80	0.20	0.12	0.73	0.61	0.59	0.40	0.97	0.86	0.97	0.88
0.72	0.86	0.25	0.12	0.78	0.50	0.65	0.60	0.97	0.59	1.00	0.92
0.65	0.86	0.22	0.12	0.71	0.57	0.57	0.44	0.76	0.50	0.72	0.60
0.68	0.77	0.25	0.08	0.75	0.73	0.70	0.72	0.87	0.73	0.83	0.92
0.65	0.84	0.22	0.16	0.77	0.64	0.58	0.52	0.71	0.64	0.71	0.80
0.58	0.82	0.14	0.08	0.80	0.57	0.71	0.60	0.97	0.64	0.97	0.80
0.65	0.75	0.25	0.20	0.77	0.64	0.68	0.48	0.80	0.61	0.71	0.72
0.61	0.77	0.28	0.08	0.81	0.66	0.62	0.52	0.85	0.75	0.83	0.84

Avg and std of each column

	Setosa				Versicolor				Virginica			
	x1	x2	x3	x4	x1	x2	x3	x4	x1	x2	x3	x4
	0.56	0.66	0.2	0.08	0.84	0.66	0.67	0.52	0.85	0.57	0.84	0.72
	0.62	0.7	0.22	0.04	0.66	0.61	0.57	0.56	0.91	0.82	0.88	1
	0.68	0.84	0.22	0.08	0.63	0.45	0.51	0.4	0.82	0.73	0.74	0.8
	0.61	0.77	0.23	0.08	0.75	0.68	0.61	0.6	0.81	0.61	0.77	0.76
	0.61	0.68	0.2	0.04	0.76	0.5	0.58	0.4	0.86	0.68	0.8	0.84
	0.54	0.68	0.16	0.04	0.77	0.66	0.68	0.56	0.72	0.57	0.72	0.8
	0.73	0.91	0.17	0.08	0.71	0.66	0.52	0.52	0.73	0.64	0.74	0.96
	0.72	1	0.22	0.16	0.85	0.7	0.64	0.56	0.81	0.73	0.77	0.92
	0.68	0.89	0.19	0.16	0.71	0.68	0.65	0.6	0.82	0.68	0.8	0.72
	0.65	0.8	0.2	0.12	0.73	0.61	0.59	0.4	0.97	0.86	0.97	0.88
	0.72	0.86	0.25	0.12	0.78	0.5	0.65	0.6	0.97	0.59	1	0.92
	0.65	0.86	0.22	0.12	0.71	0.57	0.57	0.44	0.76	0.5	0.72	0.6
	0.68	0.77	0.25	0.08	0.75	0.73	0.7	0.72	0.87	0.73	0.83	0.92
	0.65	0.84	0.22	0.16	0.77	0.64	0.58	0.52	0.71	0.64	0.71	0.8
	0.58	0.82	0.14	0.08	0.8	0.57	0.71	0.6	0.97	0.64	0.97	0.8
	0.65	0.75	0.25	0.2	0.77	0.64	0.68	0.48	0.8	0.61	0.71	0.72
	0.61	0.77	0.28	0.08	0.81	0.66	0.62	0.52	0.85	0.75	0.83	0.84
Avg:	0.64	0.80	0.21	0.10	0.75	0.62	0.62	0.53	0.84	0.67	0.81	0.82
Deviation:	0.04	0.07	0.03	0.04	0.04	0.06	0.05	0.07	0.07	0.08	0.08	0.08

added by Evgene Borisiuk (on 05 February 2019)

Iris Flower Database to validate

Setosa				Versicolor				Virginica			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.65	0.80	0.20	0.08	0.89	0.73	0.68	0.56	0.80	0.75	0.87	1.00
0.62	0.68	0.20	0.08	0.81	0.73	0.65	0.60	0.73	0.61	0.74	0.76
0.59	0.73	0.19	0.08	0.87	0.70	0.71	0.60	0.90	0.68	0.86	0.84
0.58	0.70	0.22	0.08	0.70	0.52	0.58	0.52	0.80	0.66	0.81	0.72
0.63	0.82	0.20	0.08	0.82	0.64	0.67	0.60	0.82	0.68	0.84	0.88
0.68	0.89	0.25	0.16	0.72	0.64	0.65	0.52	0.96	0.68	0.96	0.84
0.58	0.77	0.20	0.12	0.80	0.75	0.68	0.64	0.62	0.57	0.65	0.68
0.63	0.77	0.22	0.08	0.62	0.55	0.48	0.40	0.92	0.66	0.91	0.72

Wine dataset to design rules

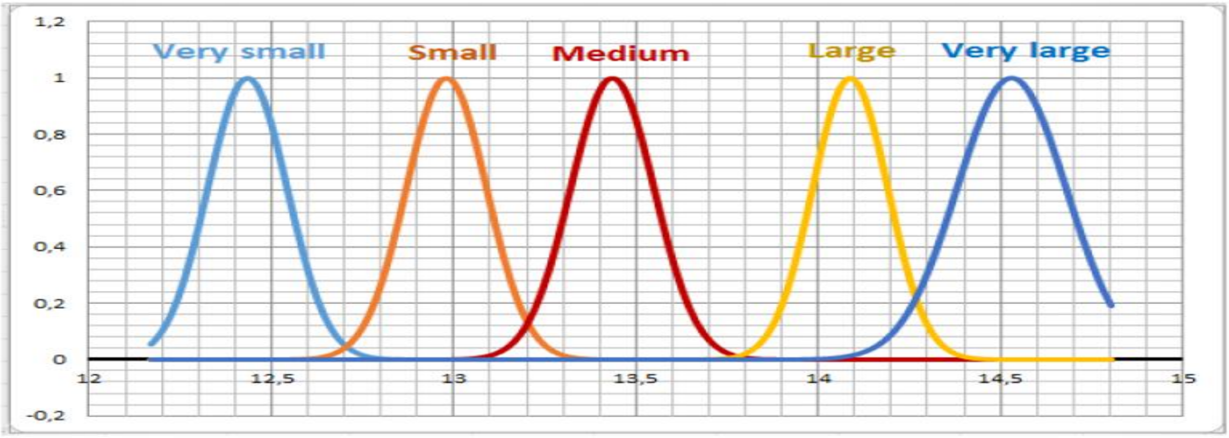
class	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13
1	14,23	1,71	2,43	15,6	127	2,8	3,06	0,28	2,29	5,64	1,04	3,92	1065
	13,2	1,78	2,14	11,2	100	2,65	2,76	0,26	1,28	4,38	1,05	3,4	1050
	13,16	2,36	2,67	18,6	101	2,8	3,24	0,3	2,81	5,68	1,03	3,17	1185
	14,37	1,95	2,5	16,8	113	3,85	3,49	0,24	2,18	7,8	0,86	3,45	1480
	13,24	2,59	2,87	21	118	2,8	2,69	0,39	1,82	4,32	1,04	2,93	735
	14,2	1,76	2,45	15,2	112	3,27	3,39	0,34	1,97	6,75	1,05	2,85	1450
	14,39	1,87	2,45	14,6	96	2,5	2,52	0,3	1,98	5,25	1,02	3,58	1290
	14,06	2,15	2,61	17,6	121	2,6	2,51	0,31	1,25	5,05	1,06	3,58	1295
	14,83	1,64	2,17	14	97	2,8	2,98	0,29	1,98	5,2	1,08	2,85	1045
	13,86	1,35	2,27	16	98	2,98	3,15	0,22	1,85	7,22	1,01	3,55	1045
2	12,37	0,94	1,36	10,6	88	1,98	0,57	0,28	0,42	1,95	1,05	1,82	520
	12,33	1,1	2,28	16	101	2,05	1,09	0,63	0,41	3,27	1,25	1,67	680
	12,64	1,36	2,02	16,8	100	2,02	1,41	0,53	0,62	5,75	0,98	1,59	450
	13,67	1,25	1,92	18	94	2,1	1,79	0,32	0,73	3,8	1,23	2,46	630
	12,37	1,13	2,16	19	87	3,5	3,1	0,19	1,87	4,45	1,22	2,87	420
	12,17	1,45	2,53	19	104	1,89	1,75	0,45	1,03	2,95	1,45	2,23	355
	12,37	1,21	2,56	18,1	98	2,42	2,65	0,37	2,08	4,6	1,19	2,3	678
	13,11	1,01	1,7	15	78	2,98	3,18	0,26	2,28	5,3	1,12	3,18	502
	12,37	1,17	1,92	19,6	78	2,11	2	0,27	1,04	4,68	1,12	3,48	510
	13,34	0,94	2,36	17	110	2,53	1,3	0,55	0,42	3,17	1,02	1,93	750
3	12,86	1,35	2,32	18	122	1,51	1,25	0,21	0,94	4,1	0,76	1,29	630
	12,88	2,99	2,4	20	104	1,3	1,22	0,24	0,83	5,4	0,74	1,42	530
	12,81	2,31	2,4	24	98	1,15	1,09	0,27	0,83	5,7	0,66	1,36	560
	12,7	3,55	2,36	21,5	106	1,7	1,2	0,17	0,84	5	0,78	1,29	600
	12,51	1,24	2,25	17,5	85	2	0,58	0,6	1,25	5,45	0,75	1,51	650
	12,6	2,46	2,2	18,5	94	1,62	0,66	0,63	0,94	7,1	0,73	1,58	695
	12,25	4,72	2,54	21	89	1,38	0,47	0,53	0,8	3,85	0,75	1,27	720
	12,53	5,51	2,64	25	96	1,79	0,6	0,63	1,1	5	0,82	1,69	515
	13,49	3,59	2,19	19,5	88	1,62	0,48	0,58	0,88	5,7	0,81	1,82	580
	12,84	2,96	2,61	24	101	2,32	0,6	0,53	0,81	4,92	0,89	2,15	590

From the work by Savchuk Artem (2016 Fall)

Two sets of membership function from 13 attributes (1)

Membership functions for attribute x1(Alcohol):

	very small	small	medium	large	very large
average	12,43	12,98	13,435	14,0875	14,53
std	0,024	0,024	0,0263	0,0213	0,045

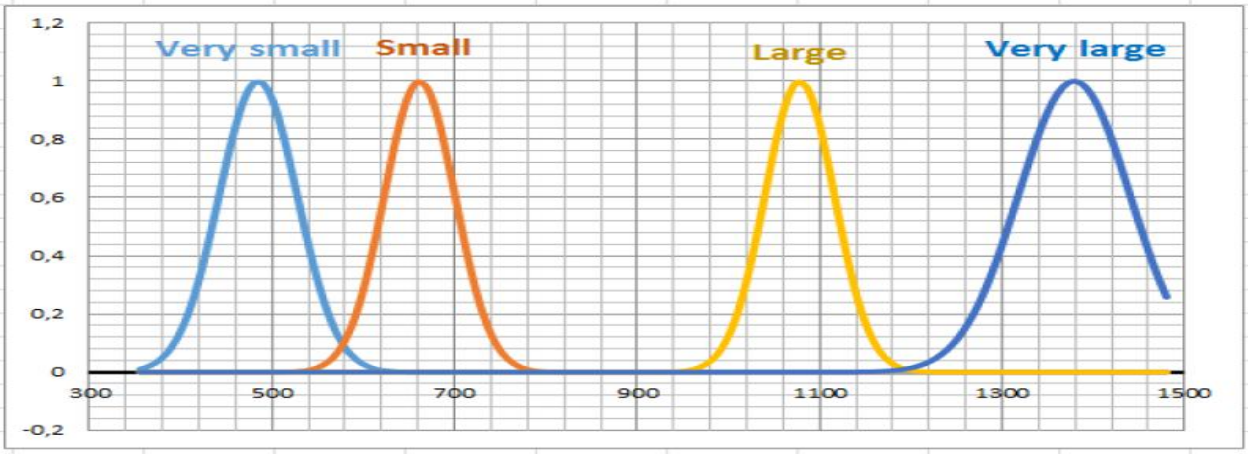


From the work by Savchuk Artem (2016 Fall)

Two sets of membership function from 13 attributes (2)

Membership functions for attribute x13 (Proline):

	very small	small	medium	large	very large
average	468,67	661,5	0	1078	1378,75
std	3670,89	3003,91	0	2916	7554,69



From the work by Savchuk Artem (2016 Fall)

Rules to classify a wine dataset

#	If X1	AND X2	AND X3	AND X4	AND X5	AND X6	AND X7	AND X8	AND X9	AND X10	AND X11	AND X12	AND X13	Then
1	large	small	large	very small	small	large	very large	small	large	very large	large	medium	very large	A
2	very large	small	large	medium	medium	medium	large	very small	large	medium	large	large	large	A
3	very small	very small	medium	very large	very small	small	medium	small	large	medium	large	very small	small	B
4	medium	very small	small	medium	medium	medium	small	large	very small	small	large	small	small	B
5	small	medium	large	large	small	small	very small	medium	small	very large	small	medium	very small	C
6	very small	small	very large	very large	large	very small	very small	large	very small	medium	small	small	small	C
7	very large	large	small	small	very large	very large	small	large	very large	large	medium	very large	medium	Other

From the work by Savchuk Artem (2016 Fall)

Wine data for validation

class	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13
1	14,1	2,16	2,3	18	105	2,95	3,32	0,22	2,38	5,75	1,25	3,17	1510
	14,12	1,48	2,32	16,8	95	2,2	2,43	0,26	1,57	5	1,17	2,82	1280
	13,75	1,73	2,41	16	89	2,6	2,76	0,29	1,81	5,6	1,15	2,9	1320
	14,75	1,73	2,39	11,4	91	3,1	3,69	0,43	2,81	5,4	1,25	2,73	1150
	14,38	1,87	2,38	12	102	3,3	3,64	0,29	2,96	7,5	1,2	3	1547
2	12,21	1,19	1,75	16,8	151	1,85	1,28	0,14	2,5	2,85	1,28	3,07	718
	12,29	1,61	2,21	20,4	103	1,1	1,02	0,37	1,46	3,05	0,906	1,82	870
	13,86	1,51	2,67	25	86	2,95	2,86	0,21	1,87	3,38	1,36	3,16	410
	13,49	1,66	2,24	24	87	1,88	1,84	0,27	1,03	3,74	0,98	2,78	472
	12,99	1,67	2,6	30	139	3,3	2,89	0,21	1,96	3,35	1,31	3,5	985
3	12,93	2,81	2,7	21	96	1,54	0,5	0,53	0,75	4,6	0,77	2,31	600
	13,36	2,56	2,35	20	89	1,4	0,5	0,37	0,64	5,6	0,7	2,47	780
	13,52	3,17	2,72	23,5	97	1,55	0,52	0,5	0,55	4,35	0,89	2,06	520
	13,62	4,95	2,35	20	92	2	0,8	0,47	1,02	4,4	0,91	2,05	550
	12,25	3,88	2,2	18,5	112	1,38	0,78	0,29	1,14	8,21	0,65	2	855

From the work by Savchuk Artem (2016 Fall)

Result of validate rules

No.	Family A	Family B	Family C	Evaluation
#1	A	B	C	Good
#2	A	C	C	Not Good
#3	A	A	C	Not Good
#4	A	B	C	Good
#5	A	Other	C	Not Good
Success rate	100%	40,00%	100%	40%

From the work by Savchuk Artem (2016 Fall)

Practice 3

To design classification rules

1. Select one dataset from those given
2. Create table with raw data from upper half of the dataset
   by sorting data in each column from the smalest to the largest
3. Each colum is devided into 5 categories: VS, S, M, L and VL by (max-min)/5
4. Define Gaussian membership functions μ_{ij} for $i = 1, 2, \ldots, N$ and $j = 1, 2, 3, 4, 5$
   where M is the number of attributes
   and
   $j = 1, 2, 3, 4, 5$ means VS, S, M, L, VL, respectively
5. Translate the table in 2. into fuzzy variables VS, S, M, L, VL
6. Create P rules from upper half of the dataset

To validate the rules

7. Apply Takagi-Sugeno formula to lower half of the dataset
8. Create table with columns being $x_1, x_2, \cdots x_N$ real class , calculated class
9. Calculate overall success rare

From the work by Savchuk Artem (2016 Fall)

IV. Time-series prediction by Fuzzy

Forecasting a value from its history

Assume $y(t)$ is a value of a variable y at time t such as maximum price of a stock during a day. Then T-S formula for singleton consequeance is as follows (Taken from Sheta, A. F. () Forecasting the Nile river flow using fuzzy logic model)

R_i: If $y(t-1)$ is A_1^i and $y(t-2)$ is A_2^i and $\cdots$ and $y(t-n+1)$ is A_n^i then $y(t)$ is g^i.

Forecasting a value from other related items

R_i: If $x_1(t)$ is A_1^i and $x_2(t)$ is A_2^i and $\cdots$ and $x_n(t)$ is A_n^i then $y(t)$ is g^i.

A stock dataset

Date	Open	Close	Date	Open	Close
9/26/2007	13779.3	13878.15	8/9/2007	13652.33	13270.68
9/25/2007	13757.84	13778.65	8/8/2007	13497.23	13657.86
9/24/2007	13821.57	13759.06	8/7/2007	13467.72	13504.3
9/21/2007	13768.33	13820.19	8/6/2007	13183.13	13468.78
9/20/2007	13813.52	13766.7	8/3/2007	13462.25	13181.91
9/19/2007	13740.61	13815.56	8/2/2007	13357.82	13463.33
9/18/2007	13403.18	13739.39	8/1/2007	13211.09	13362.37
9/17/2007	13441.95	13403.42	7/31/2007	13360.66	13211.99
9/14/2007	13421.39	13442.52	7/30/2007	13266.21	13358.31
9/13/2007	13292.38	13424.88	7/27/2007	13472.68	13265.47
9/12/2007	13298.31	13291.65	7/26/2007	13783.12	13473.57
9/11/2007	13129.4	13308.39	7/25/2007	13821.4	13785.79
9/10/2007	13116.39	13127.85	7/24/2007	13940.9	13716.95
9/7/2007	13360.74	13113.38	7/23/2007	13851.73	13943.42
9/6/2007	13306.44	13363.35	7/20/2007	14000.73	13851.08
9/5/2007	13442.85	13305.47	7/19/2007	13918.79	14000.41
9/4/2007	13358.39	13448.86	7/18/2007	13955.05	13918.22
8/31/2007	13240.84	13357.74	7/17/2007	13951.96	13971.55
8/30/2007	13287.91	13238.73	7/16/2007	13907.09	13950.98
8/29/2007	13043.07	13289.29	7/13/2007	13859.86	13907.25
8/28/2007	13318.43	13041.85	7/12/2007	13579.33	13861.73
8/27/2007	13377.16	13322.13	7/11/2007	13500.4	13577.87
8/24/2007	13231.78	13378.87	7/10/2007	13648.59	13501.7
8/23/2007	13237.27	13235.88	7/9/2007	13612.66	13649.97
8/22/2007	13088.26	13236.13	7/6/2007	13559.01	13611.68
8/21/2007	13120.05	13090.86	7/5/2007	13576.24	13565.84
8/20/2007	13078.51	13121.35	7/3/2007	13556.87	13577.3
8/17/2007	12848.05	13079.08	7/2/2007	13409.6	13535.43
8/16/2007	12859.52	12845.78	6/29/2007	13422.61	13408.62
8/15/2007	13021.93	12861.47	6/28/2007	13427.48	13422.28
8/14/2007	13235.72	13028.92	6/27/2007	13336.93	13427.73
8/13/2007	13238.24	13236.53	6/26/2007	13352.37	13337.66

V. Fuzzy Clustering

Fuzzy Relation

⋆ Example 4 ... X = {green, yellow, red}, Y = {unripe, semiripe, ripe}.

We may assume that a red apple is *provably* ripe, but a green apple is *most likely*, and so on. Thus, for example:

$X \setminus Y$	unripe	semiripe	ripe
green	1	0.5	0
yellow	0.3	1	0.4
red	0	0.2	1

Let's call this relation R_1. Then we think a similar but new Relation.

Combine two fuzzy relations

Now

$$Y = \{unripe, semiripe, ripe\}$$

and

$$Z = \{sour, sour - sweet, sweet\}$$

Let's call this relation R_2.

$X \setminus Y$	sour	sour-sweet	sweet
unripen	0.8	0.5	0.1
semiripe	0.1	0.7	0.5
ripe	0.2	0.3	0.9

Combine two fuzzy relations - continued We combine these two relations R_1 and R_2 by the formula

$$\mu_R(x,z) \geq \max_{y \in X}\{\min\{\mu_R(x,y), \mu_R(y,z)\}\},$$

the result is:

$X \setminus Y$	sour	sour-sweet	sweet
red	0.8	0.5	0.5
yellow	0.3	0.7	0.5
green	0.2	0.3	0.9

Expression by our daily language

This relation could be expressed by our daily language like

> "If tomato is red then it's most likely sweet , possibly sour-sweet, and unlikely sour."
>
> "If tomato is yellow then probably it's sour-sweet , possibly sour, maybe sweet."
>
> "If tomato is green then almost always sour, less likely sour-sweet, unlikely sweet."

Or, we could say:

> "Now tomato is more or less red, then what is taste like?"

Clustering by Fuzzy Relation of Proximity

Algorithm 2 *0. Initialize I and C*

1. *Calculate a max-min similarity-relation $R^{(0)} = [a_{ij}]$*
2. *Set $a_{ij} = 0$ for all $a_{ij} < \alpha$ and $i = j$*
3. *Select s and t such that $a_{st} = \max\{a_{ij} | i < j AND i, j \in I\}$. When tie, select one of these pairs at random*

   WHILE $a_{st} \neq 0$ DO put s and t into the same cluster $C = \{s, t\}$ ELSE 4.
   ELSE all indices $\in I$ into separated clusters and STOP

4. *Choose $u \in I - C$ such that*

$$\sum_{i \in C} a_{iu} = \max_{j \in I - C} \{\sum_{i \in C} a_{ij} | a_{ij} \neq 0\}$$

   When a tie, select one such u at random.

   WHILE such a u exists, put u into $C = \{s, t, u\}$ and REPEAT 4.

5. *Let $I = I - C$ and GOTO 3.*

Initialize

$$I = \{1, 2, 3, 4, \ldots, N\}$$

and

$$C = \{\ \}$$

Find maximum a_{ij} (if some are equal, select one at random)

$$
\begin{array}{ccccccccc}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{array}
\;\rightarrow
$$

Look for

Maximum a_{ij} was a_{st}

Put s and t to C{ }

Assume now e.g. a_{47}

So

$$C = \{4, 7\}$$

$$I = \{1, 2, 3, 5, 6, 8, 9\}$$

Then calculate $\max\{\sum_{i=1}^{2}(a_{is}+a_{it})\}$

I.e. $\max\{(a_{sj}+a_{tj}),\ (a_{sj}+a_{tj}),\ \cdots,\ (a_{sj}+a_{tj})\}$

If multiple such j then select one at random

$$
\begin{matrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{matrix}
$$

Assume now e.g. $a_{94}+a_{97}$ is such maximum

Then put 9 into C

$C = \{4, 7, 9\}$

$I = \{1, 2, 3, 5, 6, 8\}$

Choose u from

such that $\sum_{i \varepsilon C} a_{iu} \max_{j \varepsilon \{I-C\}} \{ \sum_{i \varepsilon C} a_{ij} \quad | \quad a_{ij} \neq 0 \}$

If multiple such u then select one at random

$$
\begin{matrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{matrix}
$$

Assume now $a_{24} + a_{27} + a_{29}$ is such maximum

Put 2 into C

$C = \{2, 4, 7, 9\}$

$I = \{1, 3, 5, 6, 8\}$

Repeat this prosedure

$$\begin{array}{ccccccccc}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{array}$$

Assume now

$a_{11} + a_{12} + a_{14} + a_{17} + a_{19}$

is such maximum

Put 2 into C

$$C = \{1, 2, 4, 7, 9\}$$

$$I = \{3, 5, 6, 8\}$$

Till a_{ij} included is/are 0 Or no such maximum

$$\begin{array}{ccccccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} \end{array}$$

If

$$a_{11} + a_{12} + a_{14} + a_{17} + a_{19}$$

is such maximum like previous slide

but incase $a_{11} = 0$ for example

Stop

And start again from the beginning

searching for another cluster

with

$$I = \{2, 4. 7, 9\}$$

$$C = \{\ \}$$

Example: Let's Start with the following $R^{(0)}$,

$$R^{(0)} = \begin{bmatrix} 1 & .7 & .5 & .8 & .6 & .6 & .5 & .9 & .4 & .5 \\ .7 & 1 & .3 & .6 & .7 & .9 & .4 & .8 & .6 & .6 \\ .5 & .3 & 1 & .5 & .5 & .4 & .1 & .4 & .7 & .6 \\ .8 & .6 & .5 & 1 & .7 & .5 & .5 & .7 & .5 & .6 \\ .6 & .7 & .5 & .7 & 1 & .6 & .4 & .5 & .8 & .9 \\ .6 & .9 & .4 & .5 & .6 & 1 & .3 & .7 & .7 & .5 \\ .5 & .4 & .1 & .5 & .4 & .3 & 1 & .6 & .2 & .3 \\ .9 & .8 & .4 & .7 & .5 & .7 & .6 & 1 & .4 & .4 \\ .4 & .6 & .7 & .5 & .8 & .7 & .2 & .4 & 1 & .7 \\ .5 & .6 & .6 & .6 & .9 & .5 & .3 & .4 & .7 & 1 \end{bmatrix}$$

Then repeat $R^{(n+1)} = R^{(n)} \circ R^{(n)}$ till $R^{(n)} = R^{(n+1)}$.

$$R^{(n)} = \begin{bmatrix} 1 & .2 & .5 & .8 & .6 & .2 & .3 & .9 & .4 & .3 \\ .2 & 1 & .3 & .6 & .7 & .9 & .2 & .8 & .3 & .2 \\ .5 & .3 & 1 & .5 & .3 & .4 & .1 & .3 & .7 & .6 \\ .8 & .6 & .5 & 1 & .7 & .3 & .5 & .4 & .1 & .3 \\ .6 & .7 & .3 & .7 & 1 & .2 & .4 & .5 & .8 & .9 \\ .2 & .9 & .4 & .3 & .2 & .4 & .1 & .3 & .7 & .2 \\ .3 & .2 & .1 & .5 & .4 & .1 & 1 & .6 & .1 & .3 \\ .9 & .8 & .3 & .4 & .5 & .3 & .6 & 1 & 0 & .2 \\ .4 & .3 & .7 & .1 & .8 & .7 & .1 & 0 & 1 & .1 \\ .3 & .2 & .6 & .3 & .9 & .2 & .3 & .2 & .1 & 1 \end{bmatrix}$$

Now assumming $\alpha = 0.55$ apply [1.] and [2.]

$$\begin{bmatrix} 0 & .7 & 0 & .8 & .6 & .6 & 0 & .9 & 0 & 0 \\ .7 & 0 & 0 & .6 & .7 & .9 & 0 & .8 & .6 & .6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .7 & .6 \\ .8 & .6 & 0 & 0 & .7 & 0 & 0 & .7 & 0 & .6 \\ .6 & .7 & 0 & .7 & 0 & .6 & 0 & 0 & .8 & .9 \\ .6 & .9 & 0 & 0 & .6 & 0 & 0 & .7 & .7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .6 & 0 & 0 \\ .9 & .8 & 0 & .7 & 0 & .7 & .6 & 0 & 0 & 0 \\ 0 & .6 & .7 & 0 & .8 & .7 & 0 & 0 & 0 & .7 \\ 0 & .6 & .6 & .6 & .9 & 0 & 0 & 0 & .7 & 0 \end{bmatrix}$$

First, set $I = \{1,2,3,4,5,6,7,8,9,10\}$ and $C = \{\ \}$. Then

Step 3. Now $a_{18} = a_{26} = a_{5\ 10} = 0.9$ are maximum and a_{18} is randomly selected. Then $C = \{1,8\}$.

Step 4. $a_{12} + a_{82} = a_{14} + a_{84} = 1.5$ are maximum and $j = 4$ is randomly selected. Then $C = \{1,8,4\}$.

Step 4. $a_{12} + a_{42} + a_{82} = 2.1$ is maximum, then $C = \{1,8,4,2\}$.

Step 4. There are no j such that $a_{1j} + a_{2j} + a_{4j} + a_{8j}$ is maximum. Then final $C = \{1,8,4,2\}$.

- ⋆ $a_{16} + a_{26} + a_{46} + a_{86} = 0.6 + 0.9 + 0 + 0.7 = 2.2$ seems maximum but actually not because $a_{46} = 0$

  Note that $\sum_{i \in C} a_{iu} = \max_{j \in I \setminus C}\{\sum_{i \in C} a_{ij} | a_{ij} \neq 0\}$

Step 5. Let $I = \{3, 5, 6, 7, 9, 10\}$

Step 3. $a_{5\ 10} = 0.9$ is maximum. Then renew C as $\{5, 10\}$.

Step 4. $a_{59} + a_{10\ 9} = 1.5$ is maximum. Then $C = \{5, 10, 9\}$.

Step 4. There are no j in $\{3, 6, 9\}$ such that $a_{5j} + a_{9j} + a_{10j}$ is maximum. Then final $C = \{5, 10, 9\}$.

Step 5. Let $I = \{3, 6, 7\}$.

Step 3. Now $a_{36} = a_{37} = a_{67} = 0$. Then $\{3\}$, $\{6\}$and $\{7\}$are three separated clusters. In fact,

$$\begin{bmatrix} a_{33} & a_{36} & a_{37} \\ a_{63} & a_{66} & a_{67} \\ a_{73} & a_{76} & a_{77} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\sum_{i\in\{3,6,7\}} a_{iu} = \max_{j\in\{3,6,7\}}\{\sum_{i\in C} a_{ij} | a_{ij} \neq 0\}$ does not exit any more.

In this way, when $\alpha = 0.55$, we have 5 clasters $\{1, 8, 4, 2\}$,$\{5, 10, 9\}$, $\{3\}$, $\{6\}$ and $\{7\}$ are obtained.

An example (1) Russian 33 alphabets

An example (2) A set of 13 Japanese characters

	日	田	旦	貝	回	甲	口	中	目	今	由	木	林
日	1	0.4	0.7	0.6	0.3	0.3	0.3	0.3	0.8	0.1	0.3	0.1	0.1
田	0.4	1	0.3	0.3	0.3	0.9	0.3	0.8	0.3	0.1	0.9	0.1	0.1
旦	0.7	0.3	1	0.8	0.3	0.3	0.3	0.3	0.9	0.1	0.3	0.1	0.1
貝	0.6	0.3	0.8	1	0.3	0.3	0.3	0.3	0.8	0.1	0.3	0.1	0.1
回	0.3	0.3	0.3	0.3	1	0.3	0.8	0.3	0.3	0.1	0.3	0.1	0.1
甲	0.3	0.9	0.3	0.3	0.3	1	0.3	0.9	0.3	0.1	0.8	0.1	0.1
口	0.3	0.3	0.3	0.3	0.8	0.3	1	0.3	0.3	0.1	0.3	0.1	0.1
中	0.3	0.8	0.3	0.3	0.3	0.9	0.3	1	0.3	0.1	0.9	0.1	0.1
目	0.8	0.3	0.9	0.8	0.3	0.3	0.3	0.3	1	0.1	0.3	0.1	0.1
今	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	1	0.1	0.3	0.3
由	0.3	0.9	0.3	0.3	0.3	0.8	0.3	0.9	0.3	0.1	1	0.1	0.1
木	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.3	0.1	1	0.9
林	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.3	0.1	0.9	1

		1	2	3	4	5	6	7	8	9	10	11	12	13
		日	田	旦	貝	回	甲	口	中	目	今	由	木	林
1	日	0	0	0.8	0.8	0	0	0	0	0.8	0	0	0	0
2	田	0	0	0	0	0	0.9	0	0.9	0	0	0.9	0	0
3	旦	0.8	0	0	0.8	0	0	0	0	0.9	0	0	0	0
4	貝	0.8	0	0.8	0	0	0	0	0	0.8	0	0	0	0
5	回	0	0	0	0	0	0	0.8	0	0	0	0	0	0
6	甲	0	0.9	0	0	0	0	0	0.9	0	0	0.9	0	0
7	口	0	0	0	0	0.8	0	0	0	0	0	0	0	0
8	中	0	0.9	0	0	0	0.9	0	0	0	0	0.9	0	0
9	目	0.8	0	0.9	0.8	0	0	0	0	0	0	0	0	0
10	今	0	0	0	0	0	0	0	0	0	0	0	0	0
11	由	0	0.9	0	0	0	0.9	0	0.9	0	0	0	0	0
12	木	0	0	0	0	0	0	0	0	0	0	0	0	0.9
13	林	0	0	0	0	0	0	0	0	0	0	0	0.9	0

The 1st iteration

I={1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}

C={ }

$a_{26}=a_{28}=a_{2\,11}=a_{39}=a_{68}=a_{6\,11}=a_{8\,11}=a_{12\,13}=0.9$ are maximum and a_{26} is selected at random, then C={2, 6}

$a_{28}+a_{68}=a_{2\,11}+a_{6\,11}=0.9+0.9=1.8$ are maximum and j=8 is selected at random, then C={2, 6, 8}

$a_{2\,11}+a_{6\,11}+a_{8\,11}=0.9+0.9+0.9=2.7$ is maximum, then C={2, 6, 8, 11}

There are no such j, that $a_{2j}+a_{6j}+a_{8j}+a_{11j}$ is maximum, then C={2, 6, 8, 11}

The result, when α=0.55, is 5 clusters {2, 6, 8, 11}, {1, 3, 4, 9}, {12, 13}, {5, 7}, {10}

OR

{田, 甲, 中, 由}, {日, 旦, 貝, 目}, {木, 林}, {回, 口}, {今}