A FUZZY–FUZZY RELATION AND ITS APPLICATION TO THE CLUSTERING TECHNIQUE

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In this paper we have formulated a fuzzy-fuzzy relation which is an extension of the fuzzy one, and have investigated its properties. Using some characteristic property which holds in a family of special fuzzy grades, we have introduced an equivalence relation in an ordinary set. Finally we have demonstrated a classification by the equivalence relation.

1. Introduction

The fuzzy set theory, which is regarded as an extension the ordinary set theory, originated with Zadeh (1965, pp. 338–353) about 15 years ago. When the uncertainty is essentially non-probabilistic and ascribable to one's subjectivity or absence of a well-defined boundary of membership, the fuzzy set theory will be powerful. In 1975 Zadeh proposed the fuzzy-fuzzy set theory (Zadeh, 1975a, 1975b, 1975c), or more generally n-type fuzzy set, as an extension of fuzzy set, and its algebraic properties have been studied. As in as the fuzzy case (Tamura et al., 1971, pp. 61–66), we show a characteristic property of the composition of fuzzy-fuzzy relations and the product of fuzzy-fuzzy matrices. There, however, exists an essential difference between two cases. In the fuzzy case all the elements of the fuzzy matrix are comparable each other, whereas in the fuzzy-fuzzy case all the elements of the fuzzy-fuzzy matrices are not necessarily comparable each other. Finally, by using this property we show a procedure of classification of an ordinary set.

2. Fuzzy-Fuzzy set

In this section we give a brief description of fuzzy-fuzzy set. The underlying concepts of our formulation are two operations, i.e., union and intersection, and normality and convexity of fuzzy grades. According to Zadeh, a fuzzy-fuzzy set, written $A$, on a non-empty ordinary set $X$ is characterized by a membership function, denoted by $\mu_A$, defined as follows:

$$\mu_A : X \rightarrow [0, 1]^J,$$

where $J$ is any non-empty subset of the closed unit interval $[0, 1]$. In the case of fuzzy set $\mu_A(x)$ is called a grade of the element $x$. The grade $\mu_A(x)$ represents the degree to which the element $x$ enjoys the membership of the fuzzy set. The grade of member-
The element \( x \) takes a numerical value in the interval \([0, 1]\), whereas in the case of fuzzy-fuzzy sets the grade takes a function, a mapping of \( J \) into \([0, 1]\), as its value. \( \mu_A(x) \) is called a fuzzy grade of the element \( x \), which is a fuzzy set on \( J \). Furthermore, the union and the intersection of two fuzzy-fuzzy sets are defined by two binary operations \( \sqcup \) and \( \sqcap \) for their fuzzy grades. Let \( A \) and \( B \) be two fuzzy-fuzzy sets with membership functions \( \mu_A \) and \( \mu_B \), respectively. To represent fuzzy grades \( \mu_A \) and \( \mu_B \), we use the following notations:

\[
\begin{align*}
\mu_A(x) &= \sum_{u \in J} \alpha(u)/u, \\
\mu_B(x) &= \sum_{v \in J} \beta(v)/v,
\end{align*}
\]

where \( \alpha \) and \( \beta \) are two mappings of \( J \) into \([0, 1]\) and regarded as two membership functions of fuzzy sets on \( J \). The union and the intersection of \( A \) and \( B \), written \( A \cup B \) and \( A \cap B \), respectively, are defined by the following:

\[
\begin{align*}
A \cup B &\iff \mu_{A \cup B}(x) = \mu_A(x) \cup \mu_B(x) \\
&= \left( \sum_{u \in J} \alpha(u)/u \right) \cup \left( \sum_{v \in J} \beta(v)/v \right) \\
&= \sum_{(u,v) \in J \times J} \min(\alpha(u), \beta(v)) / \max(u, v), \\
A \cap B &\iff \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x) \\
&= \left( \sum_{u \in J} \alpha(u)/u \right) \sqcap \left( \sum_{v \in J} \beta(v)/v \right) \\
&= \sum_{(u,v) \in J \times J} \max(\alpha(u), \beta(v)) / \min(u, v),
\end{align*}
\]

where the symbols \( \sqcap \) and \( \sqcup \) mean min and max, respectively.

The definitions of convexity and normality are following. The fuzzy grade \( \alpha(u) \) for \( u \in J \) is called convex, if for any two elements \( u, v \in J \) (\( u < v \)) and for any element \( w \) such that \( u \leq w \leq v \), we have \( \alpha(w) \geq \alpha(u) \wedge \alpha(v) \). The fuzzy grade is called normal if the supremum of the function values \( \alpha(u) \) for \( u \in J \) is equal to unity:

\[
\sup_{u \in J} \alpha(u) = 1.
\]

### 3. Fuzzy-Fuzzy Relation

#### 3.1 Definition of Fuzzy-Fuzzy Relation

We define Fuzzy-Fuzzy relation as a natural extension of the fuzzy relation. The Fuzzy-Fuzzy relation \( R \) on two ordinary non-empty sets \( X \) and \( Y \) is defined as a fuzzy-fuzzy set on their product \( X \times Y \), and its membership function \( \mu_R \) is defined as follows:

\[
\mu_R : X \times Y \longrightarrow [0, 1]^J.
\]

When \( Y \) equals \( X \), the F-F relation, hereafter for simplicity we use the word "F-F" instead of "Fuzzy-Fuzzy", is called a binary F-F relation on \( X \). An F-F relation with convexity for the all the elements of \( X \times Y \) is called a convex F-F relation, and similarly an F-F relation with normality for all the elements of \( X \times Y \) is called a normal one.
3.2 Composition of F-F relations

Now we define a composition of two F-F relations. Let $X$, $Y$ and $Z$ be three non-empty ordinary sets. Let $R$ and $S$ be two F-F relations on $X \times Y$ and $Y \times Z$, respectively, and $\mu_R$ and $\mu_S$ their membership functions defined as follows;

$$\mu_R : X \times Y \rightarrow [0, 1]^J$$
$$\mu_S : Y \times Z \rightarrow [0, 1]^J.$$

Then the composition of $R$ and $S$, written $S \circ R$, is defined as a fuzzy-fuzzy set on the product set $X \times Z$ with the membership function $\mu_{S \circ R}$ defined by

$$\mu_{S \circ R}(x, z) = \bigvee_{y \in Y} \{\mu_R(x, y) \land \mu_S(y, z)\}.$$

3.3 Associativity of composition

Since the binary operations $\bigvee$ and $\bigwedge$ do not satisfy the distributivity for all fuzzy grades, the composition of F-F relation is not associative in general (Mizumoto et al., 1975, pp. 421-428). By assuming, however, that all fuzzy grades under consideration are convex, the family of such fuzzy grades is distributive with respect to two binary operations $\bigvee$ and $\bigwedge$, and so the composition of convex F-F relations is associative. Thus it holds that

$$(R \circ S) \circ T = R \circ (S \circ T),$$

where $R$, $S$ and $T$ are convex F-F relations and the composition results in a convex one. In particular for three F-F relations $R$, $S$ and $T$ with both normality and convexity it holds that

$$(R \circ S) \circ T = R \circ (S \circ T),$$
$$d \circ R = R \circ d,$$

where the symbol $d$ means a diagonal F-F relation defined by the following;

$$\mu_d(x, x) = 1 = 1/1, \text{ for all } x \in X;$$
$$\mu_d(x, y) = 0 = 1/0, \text{ for } x, y \in X \text{ and } x \neq y,$$

where the symbol 1 or 1/1 means a fuzzy set on $J$ whose membership function $\alpha(u)$ is defined as follows;

$$\alpha(u) = 1 \text{ for } u = 1,$$
$$\alpha(u) = 0 \text{ for } u \in J \text{ and } u \neq 1,$$

and similarly the symbol 0 or 1/0 means a fuzzy set on $J$ whose membership function $\alpha(u)$ is defined as follows;

$$\alpha(u) = 1 \text{ for } u = 0,$$
$$\alpha(u) = 0 \text{ for } u \in J \text{ and } u \neq 0.$$

It should be noticed that it is always assumed that the set $J$ contains 0 and 1.
4. Fuzzy-Fuzzy Matrix

In this section we restrict ourselves to the case of binary F-F relations on a finite set \( I = \{ I_1, I_2, \ldots, I_n \} \) and state the definition of the previous section in Fuzzy-Fuzzy matrix terms which will be defined as below. We define a Fuzzy-Fuzzy matrix, written \( M(R) \), of which the \((i, j)\)-th element has a fuzzy grade \( \mu_R(I_i, I_j) \), as a representation of a binary F-F relation \( R \) on the finite set \( I \). For simplicity we denote the element \( \mu_R(I_i, I_j) \) of an F-F matrix by \( a_{ij} \) as well as an element of an ordinary matrix, and so we may denote an F-F matrix as follows;

\[
M(R) = A = [\mu_R(I_i, I_j)] = [a_{ij}],
\]

where the subscripts \( i \) and \( j \) run from 1 to \( n \) independently, and so the F-F matrix is an \( n \)-order square matrix. Let \( A \) and \( B \) be two \( n \)-order F-F matrices. The matrix product of \( A \) and \( B \), written \( A \circ B \), is defined by

\[
C = A \circ B = [c_{ij}] = \left[ \bigcup_{k=1}^{n} (a_{ik} \cap b_{kj}) \right].
\]  

(4.1)

The matrix product \( A \circ B \) corresponds to the composition \( S \circ R \) of two binary F-F relations \( R \) and \( S \) on \( I \). We suppose that the representations of \( R \) and \( S \) are \( A = M(R) \) and \( B = M(S) \) respectively. Then we get the following:

\[
M(S \circ R) = M(R) \circ M(S) = A \circ B.
\]

As we mentioned in the previous section, since in general the composition of F-F relations is not associative, the matrix product of two F-F matrices is not so. However, if the F-F relations have convexity, then the matrix product defined by (4.1) is associative, that is, for three convex F-F matrices \( A, B \) and \( C \), it holds that

\[
(A \circ B) \circ C = A \circ (B \circ C),
\]  

(4.2)

and the matrix product results in a convex F-F matrix.

Let \( E \) be an F-F matrix representing the diagonal relation \( \Delta \) of binary F-F relations on \( I \). Then for any normal and convex F-F matrices \( A, B \) and \( C \), it holds that

\[
(A \circ B) \circ C = A \circ (B \circ C) \quad (4.3)
\]

\[
A \circ E = E \circ A.
\]

From Eq. (4.3), the family of all the \( n \)-order normal and convex matrices is a monoid under the operation of the matrix product and the identity element \( E \). In order to show a property which holds on all the normal and convex F-F matrices we introduce a partial order relation, written \( \leq \), into the family of all such matrices. For two normal and convex matrices \( A \) and \( B \), the partial order relation \( \leq \) is defined by the following;

\[
A \leq B \iff a_{ij} \leq b_{ij} \text{ for all } i \text{ and } j,
\]

where the partial order relation \( \leq \) among the elements of matrices is defined by

\[
a_{ij} \leq b_{ij} \iff a_{ij} \cap b_{ij} = a_{ij}
\]

\[
\iff a_{ij} \cap b_{ij} = b_{ij} \text{ for all } i \text{ and } j.
\]
If all the diagonal elements of an n-order convex and normal F-F matrix A have 1/1's as their fuzzy grades, then it holds that \( E \leq A \). For any convex and normal F-F matrix A it holds that
\[
E \leq A \leq A^2 \leq A^3 \leq \cdots \leq A^{n-1} = A^n = \cdots ,
\]
where \( A^k \) means the k-th power of A.

5. An Example of Classification

In this section it is shown that, using the property defined in the previous section, we can demonstrate of \( I \) into non-empty subsets, which fill out it and have no elements in common with one another. Let A be a normal and convex F-F matrix having 1/1's as its diagonal elements, and λ be a normal and convex fuzzy grade. If λ is comparable with all the elements of \( A^{n-1} \), we can define an ordinary relation \( R_\lambda \) in I defined as follows:
\[
I_i \ R_\lambda \ I_j \iff \mu_{R^{n-1}}(I_i, I_j) \geq \lambda ,
\]
where \( R^{n-1} \) is the composition of \( (n-1) \) R's. Then it is easily shown that \( R_\lambda \) is an equivalence relation in I. Each of the partition sets of such members has larger degrees than λ in the sense of the partial order relation \( \leq \), and we can consider that the members of a partition set are homogeneous or similar to each other with the stronger degrees than λ. So we can classify the set I by the equivalence relation \( R_\lambda \). We give an example of the classification by \( R_\lambda \) defined above. Let I be consisted of 16 elements, written \( I = \{I_1, I_2, \cdots , I_{16}\} \), and moreover, let each elements belong to one and only one of three disjoint groups as below:
\[
G_1 = \{I_1, I_6, I_9, I_{13}, I_{15}\} ,
G_2 = \{I_2, I_5, I_7, I_{11}, I_{14}\} ,
G_3 = \{I_3, I_4, I_8, I_{10}, I_{12}, I_{15}\} .
\]
In this example we consider a similarity relation, written R, and for the degrees of similitude we use five fuzzy grades denoted by A, B, C, D and H as below;
\[
A = 0.1/0.1+0.2/0.2+0.3/0.3+0.4/0.4+0.5/0.5+0.6/0.6+0.7/0.7+0.8/0.8+0.9/0.9+1.0/1.0
\]
\[
B = 0.1/0.1+0.2/0.2+0.3/0.3+0.4/0.4+0.5/0.5+0.6/0.6+0.7/0.7+0.8/0.8+0.9/0.9+1.0/1.0
\]
\[
C = 0.6/0.3+0.8/0.4+1.0/0.5+0.9/0.6+0.7/0.7
\]
\[
D = 0.9/0.1+1.0/0.2+0.7/0.3
\]
\[
H = 1.0/0.0+0.9/0.1+0.9/0.2+0.8/0.3+0.7/0.4 ,
\]
where \( J = \{0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\} \).

It should be noticed that these five fuzzy grades have both convexity and normality. Let \( M(R) \) be given as Fig. 1. Then we get the 15th power of \( M(R) \), written \( M(R)^{15} \), (See Fig. 2). The resulting F-F matrix \( M(R)^{15} \) has two new fuzzy grades, denoted by F and G, as below;
\[
F = 0.2/0.3+0.4/0.4+0.5/0.6+0.6/0.7+1.0/0.8+0.9/0.9+1.0/1.0
\]
\[
G = 0.9/0.1+1.0/0.2+0.8/0.3+0.7/0.4
\]
We can represent the fuzzy grades in \( M(R) \) and the resulting matrix \( M(R)^{15} \) in Hasse diagrams (See Fig. 3 and 4). Fig. 3 corresponds to \( M(R) \) and Fig. 4 to the resulting
matrix. In these diagrams, the upper the larger in the sense of the partial order relation $\leq$, and vice versa. The unconnected fuzzy grades are not comparable each other. In Fig. 3, $B$ and $C$ are not comparable each other. In Fig. 4, only the fuzzy grades enclosed by a broken line are contained in the resulting matrix $M(R)^{15}$ and all of them are comparable each other. So we may choose $C$ or $F$ as $\lambda$. When $C$ is chosen as $\lambda$, we get three partition sets as below;

$$P_1 = \{I_1, I_4, I_6, I_9, I_{10}, I_{12}, I_{13}, I_{15}, I_{18}\},$$
$$P_2 = \{I_2, I_5, I_7, I_{11}, I_{14}\},$$
$$P_3 = \{I_3\}.$$
In the case when $\lambda = F$, we have obtained four partition sets as below:
\begin{align*}
P_1 &= \{I_1, I_5, I_8, I_{10}, I_{16}\}, \\
P_2 &= \{I_2, I_5, I_8, I_{11}, I_{14}\}, \\
P_3 &= \{I_3, I_9, I_{10}, I_{12}, I_{13}\} , \\
P_4 &= \{I_3\}.
\end{align*}
In this case only one element $I_3$ is misclassified. Finally, we emphasize that even if the given F-F matrix $M(R)$ has uncomparable elements as its elements, the resulting F-F matrix $M(R)^{15}$ has comparable elements each other.

6. Conclusion

We have formulated F-F relation and F-F matrix, and explored their algebraic properties. The Fuzzy-Fuzzy set theory gives us more natural mathematical models for fuzziness, which is essentially inherent in behaviours of human beings, than the fuzzy set theory. Thus Fuzzy-Fuzzy relation defined here is a more natural mathematical model of the intrinsically qualitative criteria of human beings with respect to relations among objects, for example, similarity relation whose the degrees of similitude between two patterns depends on one's subjective decision and are non-numerical and uncomparable with each other. The Fuzzy-Fuzzy relation and matrix will be applicable to clustering technique and so on.

References


(Received May, 1980)