A slide show of our Lecture Note

Application of Fuzzy Logic

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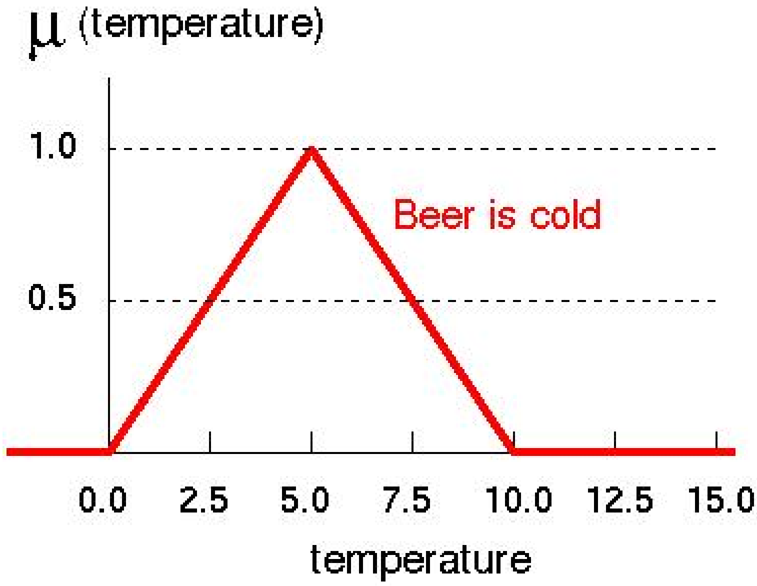
I. Fuzzy Basic Arithmetics

Membership Function

In Fuzzy logic the probability of "*how likely A is true*" is called membership value of A and expressed as μ_A. E.g., assuming A = "beer is cold," $\mu_A = 1$ when temperature of beer is 5°C, while $\mu_A = 0.5$ when temperature of beer is 10°C, and $\mu_A = 0$ when temperature of beer is 15°C.



AND and OR

Membership of A AND B and A OR B are given, respectively, as $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$

and

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

IF-THEN

Membership of IF A THEN B has proposed by many but here we use this Larsen's proposal.

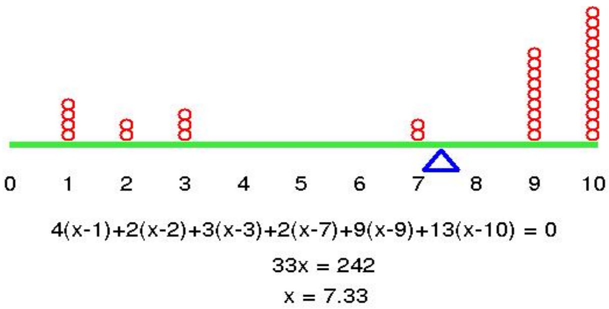
$$\mu_{A \to B}(x) = \mu_A(x) \times \mu_B(x)$$

4. De-fuzzification

When A has some different possibility, we determine most possible value of A by calculating the center of gravity of these membership values.

$$\sum_i \mu_{A_i} \times (x - x_i) = 0$$

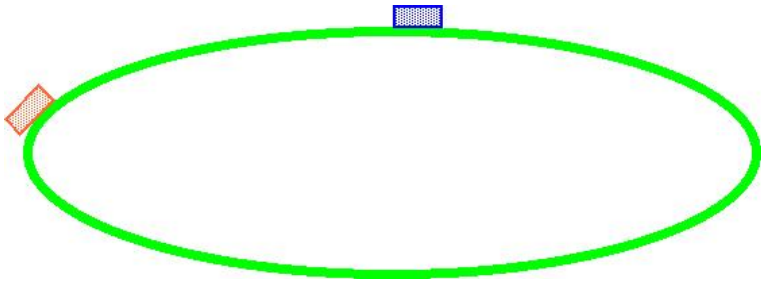
E.g.



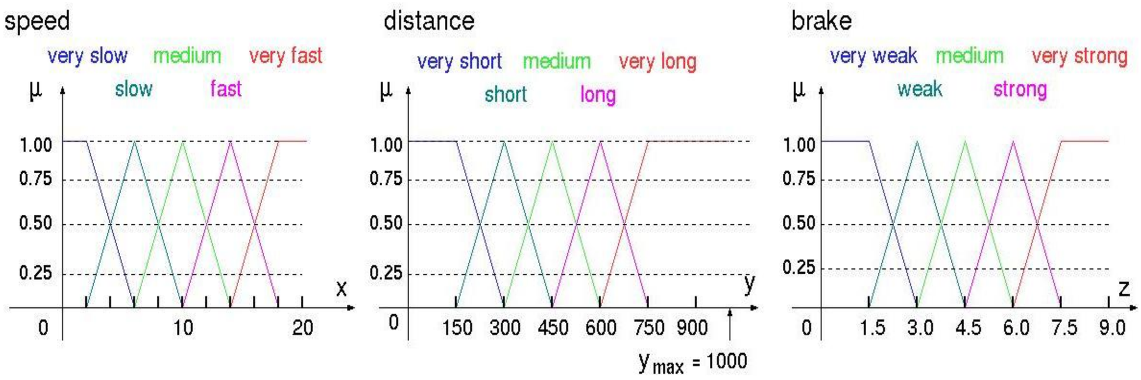
II. Fuzzy Controller

Controll two metro cars

Let's create a virtual metro system with 2 cars on a loop line with 1000 pixels. Each car has a pair of 3 parameters of speed x, distance to the car in front y and strength of brake z.



Membership function of Speed, Distance and Brake assumed here.

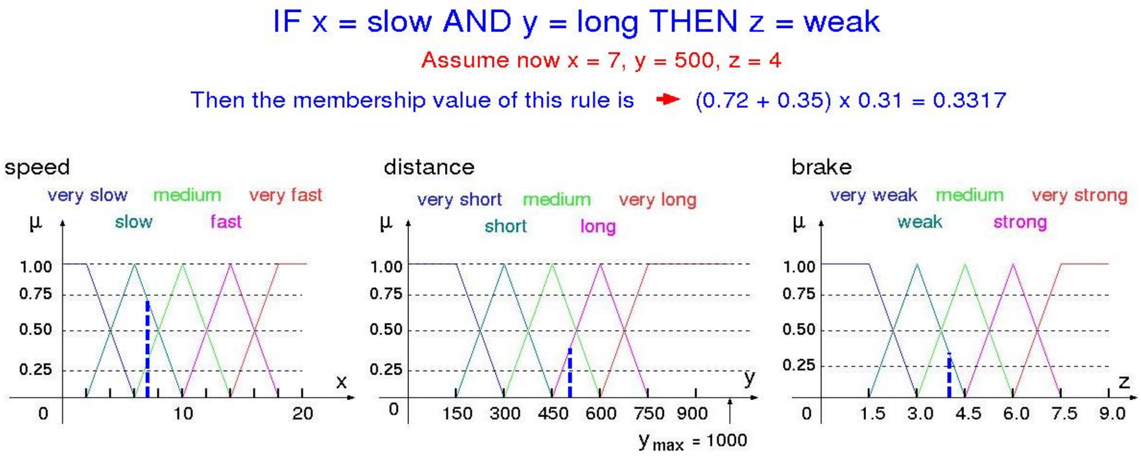


Membership value of a rule with specific speed, distance and brake.

E.g.

The membership value below implies how this brake = 4 will be likely

when speed = 7 and distance = 500 under the rule below.



An example of Membership value of one rule

Membership value of brake = 0,1,2,3,4,5,6,7,8,9 when speed = 20 and distance = 650 under the rule

IF speed = medium AND distance = long THEN brake = medium.

Rule: If speed is medium, distance is long, then break is

Speed	Mui	Distance	Mui	Break	Mui	TotalM
0	0	0	0	0	0	0
-	-	-	-	-	-	-
-	-	-	-	-	-	-
-	-	-	-	-	-	-
20	0,5	650	0,5	0	0	0
20	0,5	650	0,5	1	0	0
20	0,5	650	0,5	2	0	0
20	0,5	650	0,5	3	0	0
20	0,5	650	0,5	4	0,5	0,25
20	0,5	650	0,5	5	1	0,5
20	0,5	650	0,5	6	0,5	0,25
20	0,5	650	0,5	7	0	0
20	0,5	650	0,5	8	0	0
20	0,5	650	0,5	9	0	0
-	-	-	-	-	-	-
-	-	-	-	-	-	-
-	-	-	-	-	-	-
40	0	1000	0	9	0	0

From the work by Korol Andrey (2015 Fall)

Membership value of two rules

IF x = slow AND y = long THEN z = weak

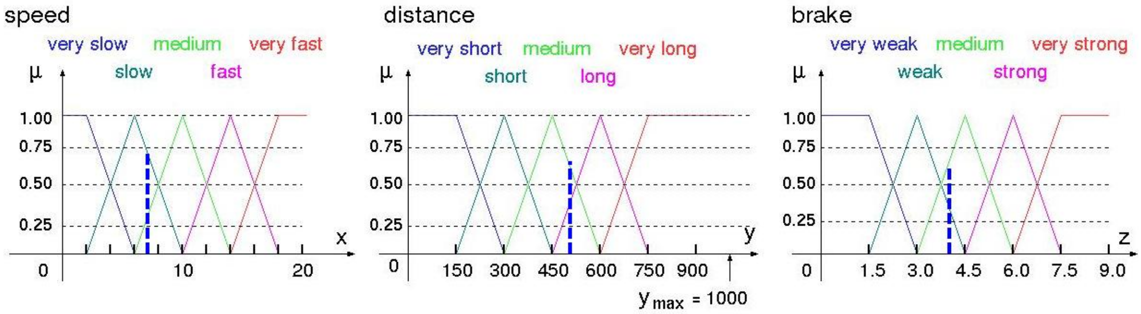
OR

IF x = medium AND y = medium THEN z = medium

Assume now x = 7, y = 500, z = 4

Then the membership value of these two rules is

max { {(0.72 + 0.35) x 0.31 = 0.3317},{(0.23 + 0.75) x 0.58 = 0.5684} } = 0.5684



Membership function of three rules for a pair of a speed and a distance

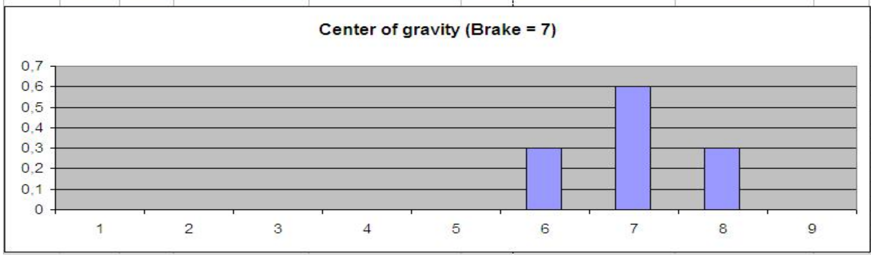
Membership value of 3 rules for a pair of speed & distance

Speed	Distance	Brake	Rule 1: IF x=medium AND y=small THEN z=strong				Rule 2: IF x=medium AND y=medium THEN z=medium				Rule 3: IF x=medium AND y=large THEN z=week				Max of rules
			mSp1	mDs1	mBr1	min(mSp;mDs)*mBr	mSp2	mDs2	mBr2	min(mSp;mDs)*mBr	mSp3	mDs3	mBr3	min(mSp;mDs)*mBr	
11,00	550,00	0	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		1	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		2	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0,25	0,1875	0,1875
		3	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	1	0,75	0,75
		4	0,75	0	0	0	0,75	0,25	0,75	0,1875	0,75	0,75	0,25	0,1875	0,1875
		5	0,75	0	0,3	0	0,75	0,25	0,75	0,1875	0,75	0,75	0	0	0,1875
		6	0,75	0	1	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		7	0,75	0	0,3	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		8	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		9	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0
		10	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0

From the work by Yulia Bogutskaya (2016 Fall)

Defuzzified value of break for a pair of a speed and a distance

			Speed is very slow AND Distance is very short THEN Brake is strong			Speed is very slow AND Distance is short THEN Brake is strong			Speed is medium AND Distance is short THEN Brake is very strong			
Speed	Distance	Brake	µ1 Speed	µ1 Distance	µ1 Brake	µ2 Speed	µ2 Distance	µ3 Brake	µ3 Speed	µ3 Distance	µ3 Brake	Result
0	150	0	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	1	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	2	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	3	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	4	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	5	1	0.4	0	1	0.6	0	0	0.6	0	0
0	150	6	1	0.4	0.5	1	0.6	0.5	0	0.6	0	0.3
0	150	7	1	0.4	1	1	0.6	1	0	0.6	0	0.6
0	150	8	1	0.4	0.5	1	0.6	0.5	0	0.6	0.5	0.3
0	150	9	1	0.4	0	1	0.6	0	0	0.6	1	0



From the work by Kuchur Alexander (2015 Fall)

Membership value of 3 rules for 3 pairs of speed & distance

Speed	Distance	Brake	Rule 1: IF x=medium AND y=small THEN z=strong				Rule 2: IF x=medium AND y=medium THEN z=medium				Rule 3: IF x=medium AND y=large THEN z=week				Max of rules	Balance
			mSp1	mDs1	mBr1	min(mSp;mDs)*mBr	mSp2	mDs2	mBr2	min(mSp;mDs)*mBr	mSp3	mDs3	mBr3	min(mSp;mDs)*mBr		
		0	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		1	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		2	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0,25	0,125	0,125	
		3	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	1	0,5	0,5	
		4	0,75	0	0	0	0,75	0,5	0,75	0,375	0,75	0,5	0,25	0,125	0,375	
11,00	500,00	5	0,75	0	0,3	0	0,75	0,5	0,75	0,375	0,75	0,5	0	0	0,375	3,727273
		6	0,75	0	1	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		7	0,75	0	0,3	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		8	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		9	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		10	0,75	0	0	0	0,75	0,5	0	0	0,75	0,5	0	0	0	
		0	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		1	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		2	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0,25	0,1875	0,1875	
		3	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	1	0,75	0,75	
		4	0,75	0	0	0	0,75	0,25	0,75	0,1875	0,75	0,75	0,25	0,1875	0,1875	
11,00	550,00	5	0,75	0	0,3	0	0,75	0,25	0,75	0,1875	0,75	0,75	0	0	0,1875	3,285714
		6	0,75	0	1	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		7	0,75	0	0,3	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		8	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		9	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		10	0,75	0	0	0	0,75	0,25	0	0	0,75	0,75	0	0	0	
		0	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		1	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		2	0,75	0	0	0	0,75	0	0	0	0,75	1	0,25	0,1875	0,1875	
		3	0,75	0	0	0	0,75	0	0	0	0,75	1	1	0,75	0,75	
		4	0,75	0	0	0	0,75	0	0,75	0	0,75	1	0,25	0,1875	0,1875	
11,00	600,00	5	0,75	0	0,3	0	0,75	0	0,75	0	0,75	1	0	0	0	3
		6	0,75	0	1	0	0,75	0	0	0	0,75	1	0	0	0	
		7	0,75	0	0,3	0	0,75	0	0	0	0,75	1	0	0	0	
		8	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		9	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	
		10	0,75	0	0	0	0,75	0	0	0	0,75	1	0	0	0	

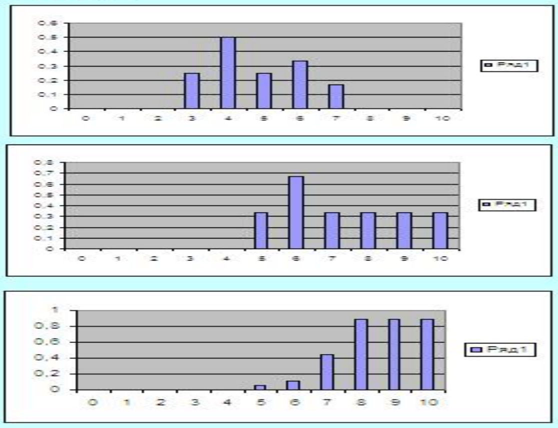
5. Membership function of 25 rules

all combination of speed = very slow ..., distance = very short

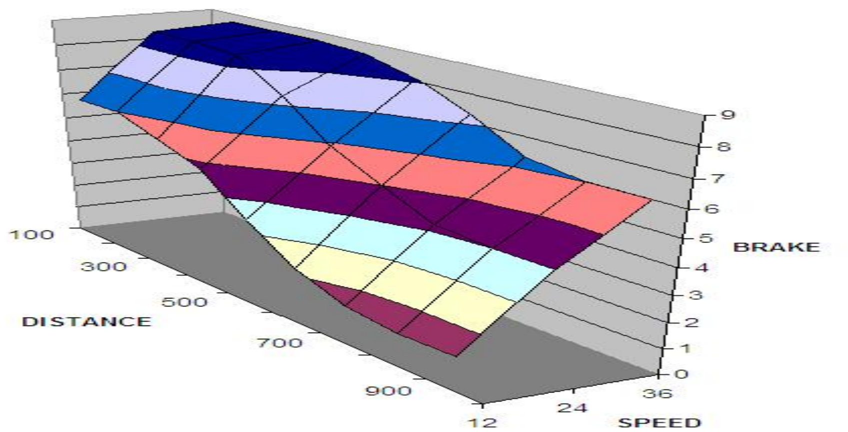
6. 3-D surface of speed-distance-brake

An example of how to draw for a fixed speed and three diferent value of distances/

Distance	Speed	Brake	Max	Center of gravity	Final brake
400	12	0	0		4,8
400	12	1	0		4,8
400	12	2	0		4,8
400	12	3	0,25		4,8
400	12	4	0,5		4,8
400	12	5	0,25		4,8
400	12	6	0,333333		4,8
400	12	7	0,166667		4,8
400	12	8	0		4,8
400	12	9	0		4,8
400	12	10	0		4,8
400	24	0	0		7,3
400	24	1	0		7,3
400	24	2	0		7,3
400	24	3	0		7,3
400	24	4	0		7,3
400	24	5	0,333333		7,3
400	24	6	0,666667		7,3
400	24	7	0,333333		7,3
400	24	8	0,333333		7,3
400	24	9	0,333333		7,3
400	24	10	0,333333		7,3
400	36	0	0		8,6
400	36	1	0		8,6
400	36	2	0		8,6
400	36	3	0		8,6
400	36	4	0		8,6
400	36	5	0,0555556		8,6
400	36	6	0,111111		8,6
400	36	7	0,444444		8,6
400	36	8	0,888889		8,6
400	36	9	0,888889		8,6
400	36	10	0,888889		8,6



Distance	Speed	Final Brake
100	12	6
100	24	8,5
100	36	8,7
200	12	6
200	24	8,5
200	36	8,7
300	12	5,4
300	24	8,4
300	36	8,7
400	12	4,8
400	24	7,3
400	36	8,6
500	12	3,4
500	24	5,9
500	36	8,1
600	12	2,2
600	24	5
600	36	7,3
700	12	1,5
700	24	4,2
700	36	6,3
800	12	1,3
800	24	4
800	36	6
900	12	1,3
900	24	4
900	36	6

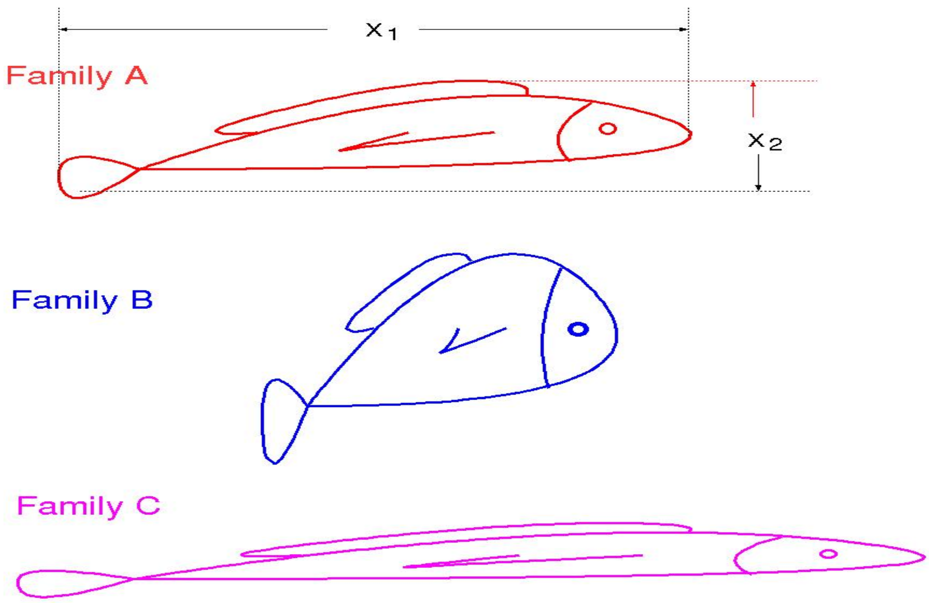


From the work by Bokhanov Evgenii (2015 Fall)

7. Control metros by 3-D surface of speed-distance-brake

III. Fuzzy Classification

An example of classification - 3 families of fish



1. Rules to classify as an example

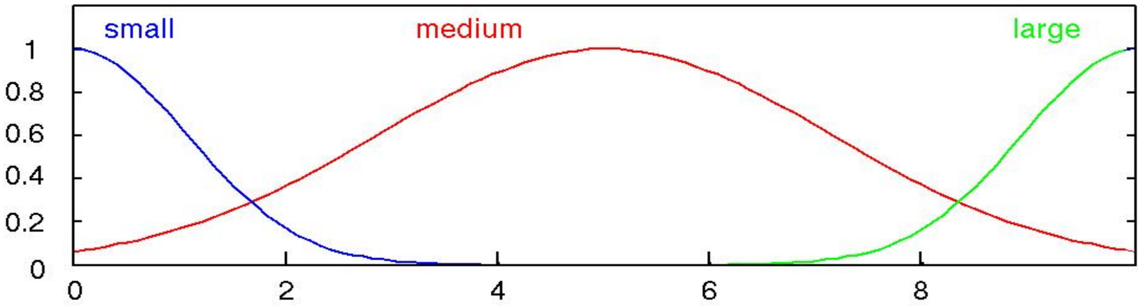
R_1: IF X_1 = medium AND X_2 = small THEN A

R_2: IF X_1 = small AND X_2 = medium THEN B

R_3: IF X_1 = large AND X_2 = small THEN C

2. Memership function for the size of two parts

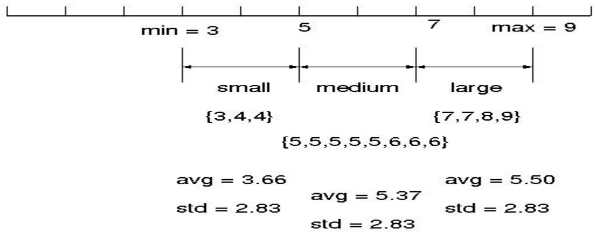
$$\mu(x) = \exp\{-\frac{(x - avg)^2}{std^2}\}$$



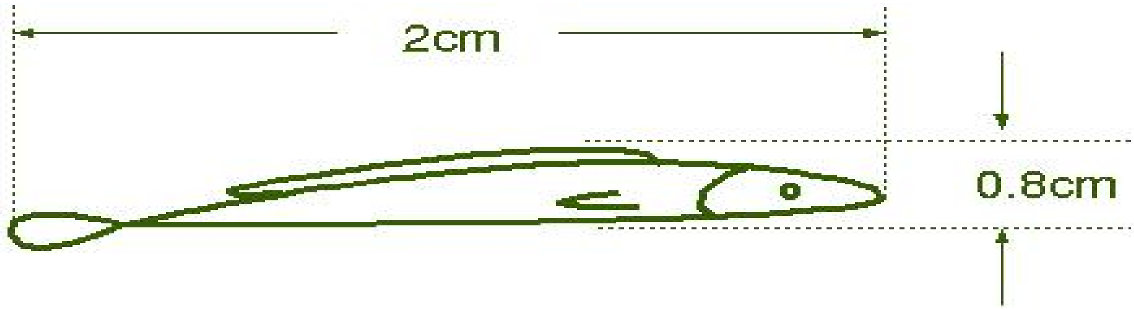
?

How we specify avg and std for each of membership function from dataset given?

ω_1			ω_2			ω_3		
X_1	X_2	X_3	X_1	X_2	X_3	X_1	X_2	X_3
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
9	9	9	9	9	9	9	9	9
5	5	5	5	5	5	5	5	5
4	4	4	4	4	4	4	4	4
3	3	3	3	3	3	3	3	3
5	5	5	5	5	5	5	5	5
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8



Question: Which family is this new fish?



2. Takagi Sugeno Formula

R_k: If x_1 is A_1^k, and x_2 is A_2^k and $\cdots$ and x_N is A_N^k then y is g^k.

Takagi-Sugeno rules: Estimation of a single input

Estimation of y for an input $\mathbf{x} = (x_1, x_2, \cdots, x_N)$

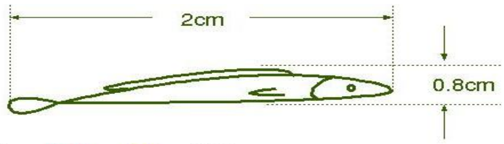
$$y_j = \frac{\sum_{k=1}^{H}(M_k(\mathbf{x}) \cdot g_k)}{\sum_{k=1}^{H} M_k(\mathbf{x})}$$

where

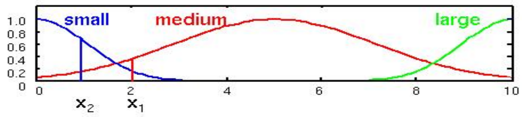
$$M_k(\mathbf{x}) = \prod_{i=1}^{N} \mu_{ik}(x_i)$$

where μ_{ik} is i-th attribute of k-th rule

Three rules to classify

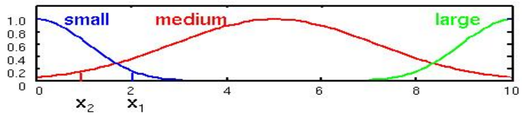


R_1: IF X1 = medium AND X2 = small THEN y = 1



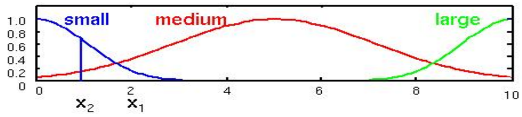
$M_1 = 0.39 \times 0.41 = 0.16$

R_2: IF X1 = small AND X2 = medium THEN y = 2



$M_2 = 0.18 \times 0.18 = 0.03$

R_3: IF X1 = large AND X2 = small THEN y = 3



$M_3 = 0.01 \times 0.71 = 0.01$

$$y = \frac{0.16 \times 1 + 0.03 \times 2 + 0.01 \times 3}{0.16 + 0.03 + 0.01} = \frac{0.25}{0.2} = 1.25$$

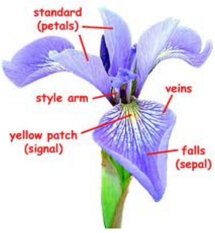
A benchmark – Iris database

Iris flower dataset (taken from University of California Urvine Machine Learning Repository) consists of three species of iris flower

setosa, versicolor and *virginica.*

Each sample represents four attributes of the iris flower

sepal-length, sepal-width, petal-length, and *petal-width.*



Iris Flower Database to design

Setosa				Versicolor				Virginica			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.56	0.66	0.20	0.08	0.84	0.66	0.67	0.52	0.85	0.57	0.84	0.72
0.62	0.70	0.22	0.04	0.66	0.61	0.57	0.56	0.91	0.82	0.88	1.00
0.68	0.84	0.22	0.08	0.63	0.45	0.51	0.40	0.82	0.73	0.74	0.80
0.61	0.77	0.23	0.08	0.75	0.68	0.61	0.60	0.81	0.61	0.77	0.76
0.61	0.68	0.20	0.04	0.76	0.50	0.58	0.40	0.86	0.68	0.80	0.84
0.54	0.68	0.16	0.04	0.77	0.66	0.68	0.56	0.72	0.57	0.72	0.80
0.73	0.91	0.17	0.08	0.71	0.66	0.52	0.52	0.73	0.64	0.74	0.96
0.72	1.00	0.22	0.16	0.85	0.70	0.64	0.56	0.81	0.73	0.77	0.92
0.68	0.89	0.19	0.16	0.71	0.68	0.65	0.60	0.82	0.68	0.80	0.72
0.65	0.80	0.20	0.12	0.73	0.61	0.59	0.40	0.97	0.86	0.97	0.88
0.72	0.86	0.25	0.12	0.78	0.50	0.65	0.60	0.97	0.59	1.00	0.92
0.65	0.86	0.22	0.12	0.71	0.57	0.57	0.44	0.76	0.50	0.72	0.60
0.68	0.77	0.25	0.08	0.75	0.73	0.70	0.72	0.87	0.73	0.83	0.92
0.65	0.84	0.22	0.16	0.77	0.64	0.58	0.52	0.71	0.64	0.71	0.80
0.58	0.82	0.14	0.08	0.80	0.57	0.71	0.60	0.97	0.64	0.97	0.80
0.65	0.75	0.25	0.20	0.77	0.64	0.68	0.48	0.80	0.61	0.71	0.72
0.61	0.77	0.28	0.08	0.81	0.66	0.62	0.52	0.85	0.75	0.83	0.84

Iris Flower Database to validate

Setosa				Versicolor				Virginica			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.65	0.80	0.20	0.08	0.89	0.73	0.68	0.56	0.80	0.75	0.87	1.00
0.62	0.68	0.20	0.08	0.81	0.73	0.65	0.60	0.73	0.61	0.74	0.76
0.59	0.73	0.19	0.08	0.87	0.70	0.71	0.60	0.90	0.68	0.86	0.84
0.58	0.70	0.22	0.08	0.70	0.52	0.58	0.52	0.80	0.66	0.81	0.72
0.63	0.82	0.20	0.08	0.82	0.64	0.67	0.60	0.82	0.68	0.84	0.88
0.68	0.89	0.25	0.16	0.72	0.64	0.65	0.52	0.96	0.68	0.96	0.84
0.58	0.77	0.20	0.12	0.80	0.75	0.68	0.64	0.62	0.57	0.65	0.68
0.63	0.77	0.22	0.08	0.62	0.55	0.48	0.40	0.92	0.66	0.91	0.72

IV. Time-series prediction by Fuzzy

Forcasting a value from its history

Assume $y(t)$ is a value of a variable y at time t such as maximum price of a stock during a day. Then T-S formula for singleton consequeance is as follows[1].

R_i: If $y(t-1)$ is A_1^i and $y(t-2)$ is A_2^i and $\cdots$ and $y(t-n+1)$ is A_n^i then $y(t)$ is g^i.

Forcasting a value from other related items

R_i: If $x_1(t)$ is A_1^i and $x_2(t)$ is A_2^i and $\cdots$ and $x_n(t)$ is A_n^i then $y(t)$ is g^i.

V. Fuzzy Clustering

1. Fuzzy Relation

⋆ Example 4 ... X = {green, yellow, red}, Y = {unripe, semiripe, ripe}.

We may assume that a red apple is *provably* ripe, but a green apple is *most likely*, and so on. Thus, for example:

$X \setminus Y$	unripe	semiripe	ripe
green	1	0.5	0
yellow	0.3	1	0.4
red	0	0.2	1

Let's call this relation R_1. Then we think a similar but new Relation.

2. Combine two fuzzy relations

Now

$$Y = \{unripe, semiripe, ripe\}$$

and

$$Z = \{sour, sour - sweet, sweet\}$$

Let's call this relation R_2.

$X \setminus Y$	sour	sour-sweet	sweet
unripen	0.8	0.5	0.1
semiripe	0.1	0.7	0.5
ripe	0.2	0.3	0.9

Combine two fuzzy relations - continued

We combine these two relations R_1 and R_2 by the formula

$$\mu_R(x, z) \geq \max_{y \in X}\{\min\{\mu_R(x, y), \mu_R(y, z)\}\},$$

the result is:

$X \setminus Y$	sour	sour-sweet	sweet
red	0.8	0.5	0.5
yellow	0.3	0.7	0.5
green	0.2	0.3	0.9

Expression by our daily language

This relation could be expressed by our daily language like

> "If tomato is red then it's most likely sweet , possibly sour-sweet, and unlikely sour."
>
> "If tomato is yellow then probably it's sour-sweet , possibly sour, maybe sweet."
>
> "If tomato is green then almost always sour, less likely sour-sweet, unlikely sweet."

Or, we could say:

> "Now tomato is more or less red, then what is taste like?"

3. Clustering by Fuzzy Relation of Proximity

Algorithm 1 *1. Calculate a max-min similarity-relation* $R = [a_{ij}]$

2. Set $a_{ij} = 0$ *for all* $a_{ij} < \alpha$ *and* $i = j$

3. Select s *and* t *such that* $a_{st} = \max\{a_{ij} | i < j \,\&\, i, j \in I\}$*. When tie, select one of these pairs at random*

WHILE $a_{st} \neq 0$ *DO put* s *and* t *into the same cluster* $C = \{s, t\}$ *ELSE 4. ELSE all indices* $\in I$ *into separated clusters and STOP*

4. Choose $u \in I - C$ *such that*

$$\sum_{i \in C} a_{iu} = \max_{j \in I - C} \{\sum_{i \in C} a_{ij} | a_{ij} \neq 0\}$$

When a tie, select one such u at random.

WHILE such a u *exists, put* u *into* $C = \{s, t, u\}$ *and REPEAT 4.*

5. Let $I = I - C$ *and GOTO 3.*

Example: Let's Start with the following $R^(0)$,

$$R^{(0)} = \begin{bmatrix} 1 & .7 & .5 & .8 & .6 & .6 & .5 & .9 & .4 & .5 \\ .7 & 1 & .3 & .6 & .7 & .9 & .4 & .8 & .6 & .6 \\ .5 & .3 & 1 & .5 & .5 & .4 & .1 & .4 & .7 & .6 \\ .8 & .6 & .5 & 1 & .7 & .5 & .5 & .7 & .5 & .6 \\ .6 & .7 & .5 & .7 & 1 & .6 & .4 & .5 & .8 & .9 \\ .6 & .9 & .4 & .5 & .6 & 1 & .3 & .7 & .7 & .5 \\ .5 & .4 & .1 & .5 & .4 & .3 & 1 & .6 & .2 & .3 \\ .9 & .8 & .4 & .7 & .5 & .7 & .6 & 1 & .4 & .4 \\ .4 & .6 & .7 & .5 & .8 & .7 & .2 & .4 & 1 & .7 \\ .5 & .6 & .6 & .6 & .9 & .5 & .3 & .4 & .7 & 1 \end{bmatrix}$$

Then repeat $R^{(n+1)} = R^{(n)} \circ R^{(n)}$ till $R^{(n)} = R^{(n+1)}$.

$$R^{(n)} = \begin{bmatrix} 1 & .2 & .5 & .8 & .6 & .2 & .3 & .9 & .4 & .3 \\ .2 & 1 & .3 & .6 & .7 & .9 & .2 & .8 & .3 & .2 \\ .5 & .3 & 1 & .5 & .3 & .4 & .1 & .3 & .7 & .6 \\ .8 & .6 & .5 & 1 & .7 & .3 & .5 & .4 & .1 & .3 \\ .6 & .7 & .3 & .7 & 1 & .2 & .4 & .5 & .8 & .9 \\ .2 & .9 & .4 & .3 & .2 & .4 & .1 & .3 & .7 & .2 \\ .3 & .2 & .1 & .5 & .4 & .1 & 1 & .6 & .1 & .3 \\ .9 & .8 & .3 & .4 & .5 & .3 & .6 & 1 & 0 & .2 \\ .4 & .3 & .7 & .1 & .8 & .7 & .1 & 0 & 1 & .1 \\ .3 & .2 & .6 & .3 & .9 & .2 & .3 & .2 & .1 & 1 \end{bmatrix}$$

Now ssumming $\alpha = 0.55$ apply [1.] and [2.]

$$\begin{bmatrix} 0 & .7 & 0 & .8 & .6 & .6 & 0 & .9 & 0 & 0 \\ .7 & 0 & 0 & .6 & .7 & .9 & 0 & .8 & .6 & .6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .7 & .6 \\ .8 & .6 & 0 & 0 & .7 & 0 & 0 & .7 & 0 & .6 \\ .6 & .7 & 0 & .7 & 0 & .6 & 0 & 0 & .8 & .9 \\ .6 & .9 & 0 & 0 & .6 & 0 & 0 & .7 & .7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .6 & 0 & 0 \\ .9 & .8 & 0 & .7 & 0 & .7 & .6 & 0 & 0 & 0 \\ 0 & .6 & .7 & 0 & .8 & .7 & 0 & 0 & 0 & .7 \\ 0 & .6 & .6 & .6 & .9 & 0 & 0 & 0 & .7 & 0 \end{bmatrix}$$

First, set $I = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $C = \{\ \}$. Then

3. Now $a_{18} = a_{26} = a_{5\ 10} = 0.9$ are maximum and a_{18} is randomly selected. Then $C = \{1, 8\}$.
4. $a_{12} + a_{82} = a_{14} + a_{84} = 1.5$ are maximum and $j = 4$ is randomly selected. Then $C = \{1, 8, 4\}$.
4. $a_{12} + a_{42} + a_{82} = 2.1$ is maximum, then $C = \{1, 8, 4, 2\}$.
4. There are no j such that $a_{1j} + a_{2j} + a_{4j} + a_{8j}$ is maximum. Then final $C = \{1, 8, 4, 2\}$.

   ⋆ $a_{16} + a_{26} + a_{46} + a_{86} = 0.6 + 0.9 + 0 + 0.7 = 2.2$ seems maximum but actually not because $a_{46} = 0$

   Note that $\sum_{i \in C} a_{iu} = \max_{j \in I \setminus C} \{\sum_{i \in C} a_{ij} | a_{ij} \neq 0\}$

5. Let $I = \{3, 5, 6, 7, 9, 10\}$
3. $a_{5\ 10} = 0.9$ is maximum. Then renew C as $\{5, 10\}$.
4. $a_{59} + a_{10\ 9} = 1.5$ is maximum. Then $C = \{5, 10, 9\}$.
4. There are no j in $\{3, 6, 9\}$ such that $a_{5j} + a_{9j} + a_{10j}$ is maximum. Then final $C = \{5, 10, 9\}$.
5. Let $I = \{3, 6, 7\}$.
3. Now $a_{36} = a_{37} = a_{67} = 0$. Then $\{3\}$, $\{6\}$and $\{7\}$are three separated clusters. In fact,

$$\begin{bmatrix} a_{33} & a_{36} & a_{37} \\ a_{63} & a_{66} & a_{67} \\ a_{73} & a_{76} & a_{77} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\sum_{i\in\{3,6,7\}} a_{iu} = \max_{j\in\{3,6,7\}}\{\sum_{i\in C} a_{ij} | a_{ij} \neq 0\}$ does not exit any more.

In this way, when $\alpha = 0.55$, we have 5 clasters $\{1, 8, 4, 2\}$,$\{5, 10, 9\}$, $\{3\}$, $\{6\}$ and $\{7\}$ are obtained.