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Handling forecasting problems using fuzzy time series

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Abstract

In [6–9], Song et al. proposed fuzzy time-series models to deal with forecasting problems. In [10], Sullivan and Woodall reviewed the first-order time-invariant fuzzy time series model and the first-order time-variant model proposed by Song and Chissom [6–8], where the models are compared with each other and with a time-invariant Markov model using linguistic labels with probability distributions. In this paper, we propose a new method to forecast university enrollments, where the historical enrollments of the University of Alabama shown in [7, 8] are used to illustrate the forecasting process. The average forecasting errors and the time complexity of these methods are compared. The proposed method is more efficient than the ones presented in [7, 8, 10] due to the fact that the proposed method simplifies the arithmetic operation process. Furthermore, the average forecasting error of the proposed method is smaller than the ones presented in [2, 7, 8]. © 1998 Elsevier Science B.V. All rights reserved

Keywords: Fuzzy time series; Fuzzy sets; Linguistic variable; Markov model; Time-variant model; Time-invariant model

1. Introduction

Forecasting activities play an important role in our daily life. We often forecast the weather, earthquakes, stock market, and everything which people want to foresee. People can plan or prevent beforehand by forecasting activities. It is impossible to make a one hundred percent forecast, but we can do our best to increase the accuracy of forecasts. Traditional forecasting methods can deal with many forecasting cases, but they cannot solve forecasting problems in which the historical data are linguistic values. In [6–8], Song and Chissom have proposed a first-order time-invariant fuzzy time-series model and a first-order time-variant fuzzy time-series model to solve the forecasting problems. They forecast the enrollments of the University of Alabama by using 20 years of historical enrollment data. The average forecasting errors are 3.18% in [7] and range from 4.49% down to 3.15% in [8], and it seems that the forecasting results are better than those which use traditional models to forecast.

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In [7, 8], Song and Chissom used the following method to forecast the enrollments of the University of Alabama:

$$A_i = A_{i-1} \circ R, \tag{1}$$

where A_{i-1} is the enrollment of year $i-1$, A_i is the forecasted enrollment of year i in terms of fuzzy sets, R is the union of fuzzy relations and "$\circ$" is the Max–Min composition operator. The method for deriving R presented in [7] is not the same as the one presented in [8]. In [7], the time-invariant method will take much time to compute the fuzzy relation R, but it is computed only once. In [8], the time-variant method will take less time to compute the fuzzy relation R, but it must recalculate the fuzzy relation R when forecasting the enrollments of different years. It will take a large amount of time to calculate the Max–Min operation when the fuzzy relation R is very big. The time complexity of a Max–Min operation is $O(kn^2)$, where k is the number of fuzzy logical relationships and n is the number of elements in the universe of discourse. There are some other researchers [2, 10], who proposed other methods based on [7, 8] to improve the average forecasting errors or increase the speed of calculation. In [10], Sullivan et al. reviewed the first-order time-invariant fuzzy time-series model and the first-order time-invariant model proposed by Song and Chissom, where these models are compared with each other and with a time-variant Markov model using linguistic labels with probability distributions. In [2], Chen presented a new method to forecast the university enrollments based on fuzzy time series, where the method is more efficient than [7, 8] due to the fact that it used simplified arithmetic calculations rather than the complicated Max–Min composition operations described in [7, 8], and the average forecasting error is 3.22% which is almost the same as the one presented in [7].

In this paper, we present a new method based on time-variant fuzzy time series to deal with the forecasting problems, where the historical enrollment data shown in [7, 8] are used to illustrate the forecasting process. The concept of the proposed method is that the variation of enrollment of this year is related to the trend of the enrollments of the past years. For example, if the trend of the enrollments of the past years is increasing, then the number of enrollment of this year might increase. To define the degree of variations, we perform systematic calculations to calculate the relation between the variations of last year and the other past years. Then, we can get the forecasting enrollments from the derived relation. The variation of the enrollments of last year is a criterion to forecast the enrollments of the next year due to the fact that we consider that the variation of this year is the most similar to the variation of last year. The average forecasting errors and the time complexity of the various forecasting methods are compared. The proposed method is more efficient than the ones presented in [7, 8, 10] due to the fact that the proposed method simplify the arithmetic operation process. Furthermore, the average forecasting error of the proposed method is smaller than the ones presented in [2, 7, 8].

The rest of this paper is organized as follows. In Section 2, we review the concepts of fuzzy time series from [7–9]. Furthermore, we also review the first-order time-variant model and the Markov model from [8, 10]. In Section 3, we present a new method for forecasting enrollments based on fuzzy time series. In Section 4, we compare the time complexity and the average forecasting errors between the proposed method and the ones presented in [2, 7, 8, 10]. The conclusions are discussed in Section 5.

2. Fuzzy time-series concepts

In this section, we review the concept of fuzzy time series and the forecasting methods presented in [6–8, 10]. Let U be the universe of discourse, $U = \{u_1, u_2, \ldots, u_n\}$. A fuzzy set [12] A of U is defined by

$$A = \mu_A(u_1)/u_1 + \mu_A(u_2)/u_2 + \cdots + \mu_A(u_n)/u_n, \tag{2}$$

where μ_A is the membership function of A, $\mu_A: U \to [0, 1]$, $\mu_A(u_i)$ denotes the grade of membership of u_i in A, $\mu_A(u_i) \in [0, 1]$, the symbol "/" separates the membership grades from the elements in the universe of discourse U, and the symbol " + " means "union" rather than the commonly used algebraic symbol of summation. Let $Y(t)$ $(t = \dots, 0, 1, 2, \dots)$, a subset of R^1, be the universe of discourse on which fuzzy set $\mu_i(t)$ $(i = 1, 2, \dots)$ are defined and let $F(t)$ be a collection of $\mu_i(t)$ $(i = 1, 2, \dots)$. Then, $F(t)$ is called a fuzzy time series on $Y(t)$ $(t = \dots, 0, 1, 2, \dots)$. It is obvious that $F(t)$ can be regarded as a linguistic variable [14], and $\mu_i(t)$ can be viewed as possible linguistic values of $F(t)$, where $\mu_i(t)$ $(i = 1, 2, \dots)$ are represented by fuzzy sets. Furthermore, we can also see that $F(t)$ is a function of time t, i.e., the values of $F(t)$ can be different at different times. According to [7], if $F(t)$ is caused by $F(t-1)$ only, then this relationship is represented by

$$F(t-1) \to F(t).$$

Let $F(t)$ be a fuzzy time series. If for any time t, $F(t) = F(t-1)$ and $F(t)$ only has finite elements, then $F(t)$ is called a time-invariant fuzzy time series. Otherwise, it is called a time-variant fuzzy time series.

In [8], Song and Chissom proposed the time-variant fuzzy time-series model and forecasted the enrollments of the University of Alabama based on the time-variant fuzzy time-series model. The Song–Chissom's method for forecasting the enrollments of the University of Alabama is briefly reviewed as follows:

Step 1: Define the universe of discourse U within which fuzzy sets will be defined.

Step 2: Partition the universe of discourse U into several equal length intervals.

Step 3: Determine some linguistic values.

Step 4: Fuzzify the historical enrollment data.

Step 5: Choose a suitable parameter w, where $w > 1$, calculate the fuzzy operation $R^w(t, t-1)$ and forecast the enrollment. The time-variant fuzzy time series method can be expressed as

$$F(t) = F(t-1) \circ R^w(t, t-1), \tag{3}$$

where

$$R^w(t, t-1) = F^T(t-2) \times F(t-1) \cup F^T(t-3) \times F(t-2) \cup \cdots \cup F^T(t-w) \times F(t-w+1), \tag{4}$$

where w is called the model basis [8] denoting the number of years before t, $F(t)$ is the value of the fuzzy time series, " $\times$ " is the Cartesian product operator, and T is the transpose operator.

Step 6: Defuzzify the forecasted outputs. In [8], they use the method of neural nets to defuzzify the forecasted outputs.

In [10], Sullivan and Woodall use the Markov model to forecast the enrollment of the students of Alabama. They use the following model to deal with the forecasting problems:

$$P'_{t+1} = P'_t * R_m, \tag{5}$$

where P_t is the vector of state probabilities at time t, P_{t+1} is the vector of state probabilities at time $(t+1)$, and R_m is the transition matrix. Unlike the Max–Min composition operator used in [7, 8], the multiplication in formula (5) is a conventional matrix multiplication. Because the R_m in formula (5) does not change with time, so it is the time-invariant fuzzy time-series model. Another style of the Markov model is as follows:

$$P'_{t+1} = P'_t * R^k_m, \quad k = 1, 2, \dots, \tag{6}$$

where R^k_m varies with time, and it is a time-variant fuzzy time-series model. For more details, please refer to [10].

3. A new method to handle forecasting problems based on fuzzy time series

In this section, we present a new method to deal with the forecasting problems. Assume that the enrollment of year t is x and assume that the enrollment of year $t-1$ is y, then the variation of the enrollments between year t and year $t-1$ is equal to $x-y$. Firstly, we describe some heuristic rules which are similar to the human thought:

Rule 1: The variation of the enrollments between this year and last year is related to the variations of the enrollments between this year and the past years, and the relationship of the enrollments between this year and last year is closer than the one between this year and the other past years.

Rule 2: If the trend of the number of enrollments of the past years is increasing, then the number of enrollments of this year is increasing. If the trend of the number of enrollments of the past years is decreasing, then the number of enrollments of this year is decreasing.

From Rules 1 and 2, we might have two problems. Firstly, if the trend of the variations of the enrollments of the past years is not so obvious, how can we know the trend of the variation of the enrollment this year? Secondly, how to define the degree of variation of this year? The solutions of these two problems are described by the following heuristic rule:

Rule 3: Let the variation of last year be a criterion. Compute the fuzzy relationships between last year and the other past years based on data variations. From the derived fuzzy relationships, we can know the degrees of relationships between the variation of last year and the variations of other past years. The variation of this year can be obtained from the derived fuzzy relationships.

Based on these heuristic rules, firstly we can fuzzify the historical enrollment data. In [7, 8], Song et al., used the linguistic values (not many), (not too many), (many), (many many), (very many), (too many), (too many many) to describe the enrollments of the historical data. In this paper, we use the fuzzified variation of the historical enrollments and the linguistic values (big decrease), (decrease), (no change), (increase), (big increase), (too big increase) to forecast the university enrollments. The fuzzified variation of the historical enrollments between year t and year $t-1$ can be described as follows:

$$F(t) = u_1/(\text{big decrease}) + u_2/(\text{decrease}) + \cdots + u_i/(L) + \cdots + u_m/(\text{too big increase}), \tag{7}$$

where $F(t)$ denotes the fuzzified variation of the enrollments between year t and year $t-1$, u_i is the grade of membership to the linguistic value L, m is the number of the elements in the universe of discourse, and $1 \leqslant i \leqslant m$.

To forecast the enrollment of year t, we must decide how many years of the enrollments data will be used, where the number of years of the enrollments data we used is called the *window basis*. Suppose we set a window basis to w years, then the variation of last year is used to be a criterion and the other variations of w past years are used to form a matrix which is called the operation matrix. The criterion matrix $C(t)$ and the operation matrix $O^w(t)$ at year t are expressed as follows:

$$C(t) = F(t-1) = \begin{array}{c} \\ [\end{array} \begin{array}{cccc} (\text{big decrease}) & (\text{decrease}) & \cdots & (\text{too big increase}) \\ C_1 & C_2 & \cdots & C_m \end{array} \begin{array}{c} \\ ], \end{array} \tag{8}$$

$$O^w(t) = \begin{bmatrix} F(t-2) \\ F(t-3) \\ \vdots \\ F(t-w-1) \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} (\text{big decrease}) & (\text{decrease}) & \cdots & (\text{too big increase}) \end{array} \\ \begin{bmatrix} O_{11} & O_{12} & \cdots & O_{1m} \\ O_{21} & O_{22} & \cdots & O_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ O_{w1} & O_{w2} & \cdots & O_{wm} \end{bmatrix} \end{array}. \tag{9}$$

We can calculate the relation between the operation matrix $O^w(t)$ and the criterion matrix $C(t)$, and we can get a relation matrix $R(t)[w, m]$ by performing $R(t) = O^w(t) \otimes C(t)$, where

$$R(t) = \begin{bmatrix} O_{11} \times C_1 & O_{12} \times C_2 & \cdots & O_{1m} \times C_m \\ O_{21} \times C_1 & O_{22} \times C_2 & \cdots & O_{2m} \times C_m \\ \vdots & \vdots & \vdots & \vdots \\ O_{w1} \times C_1 & O_{w2} \times C_2 & \cdots & O_{wm} \times C_m \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ R_{w1} & R_{w2} & \cdots & R_{wm} \end{bmatrix}, \tag{10}$$

where $R_{ij} = O_{ij} \times c_j$, $1 \leqslant i \leqslant w$, $1 \leqslant j \leqslant m$, and " × " is the multiplication operation. From the relation matrix $R(t)$, we can know the degree of relationships between last year and the other past years in data variations. Then, we can get the forecasting variation of the enrollment of year t, where

$$F(t) = [\mathrm{Max}(R_{11}, R_{21}, \ldots, R_{w1}) \ \ldots \ \mathrm{Max}(R_{12}, R_{22}, \ldots, R_{w2}) \ \ldots \ \mathrm{Max}(R_{1m}, R_{2m}, \ldots, R_{wm})]. \tag{11}$$

The proposed method is now presented as follows:

Step 1: From the historical enrollment data shown in [7, 8], compute the variations of the enrollments between any two continuous years. The variation of this year is the enrollment of this year minus the enrollment of last year. For example, if the enrollment of 1972 is 13 563, and the enrollment of 1971 is 13 055, then the variation of year 1972 = 13 563 − 13 055 = 508. Based on the historical enrollment data shown in [7, 8], we can obtain the variations of the enrollments between any two continuous years as shown in Table 1. We can find the minimum increase $D_{\min}$ and maximum increase $D_{\max}$. Then we define the universe of discourse U, $U = [D_{\min} - D_1, D_{\max} + D_2]$, where D_1 and D_2 are suitable positive numbers. In this paper, we set $D_{\min} = -955$, $D_{\max} = 1291$, $D_1 = 45$, $D_2 = 109$, so U can be represented as $U = [-1000, 1400]$.

Table 1
Actual enrollments and variations of historical data

Years	Actual enrollments	Variations
1971	13055	
1972	13563	+ 508
1973	13867	+ 304
1974	14696	+ 829
1975	15460	+ 764
1976	15311	− 149
1977	15603	+ 292
1978	15861	+ 258
1979	16807	+ 946
1980	16919	+ 112
1981	16388	− 531
1982	15433	− 955
1983	15497	+ 64
1984	15145	− 352
1985	15163	+ 18
1986	15984	+ 821
1987	16859	+ 875
1988	18150	+ 1291
1989	18970	+ 820
1990	19328	+ 358
1991	19337	+ 9
1992	18876	− 461

Step 2: Partition the universe of discourse U into several even length intervals $u_1, u_2, \ldots, u_m$. In this paper, we partition the universe of discourse U into six intervals, where $u_1 = [-1000, -600]$, $u_2 = [-600, -200]$, $u_3 = [-200, 200]$, $u_4 = [200, 600]$, $u_5 = [600, 1000]$, and $u_6 = [1000, 1400]$.

Step 3: Define fuzzy sets on the universe of discourse U. First, determine some linguistic values represented by fuzzy sets to describe the degree of variation between two continuous years. In this paper, we consider six fuzzy sets which are A_1 = (big decrease), A_2 = (decrease), A_3 = (no change), A_4 = (increase), A_5 = (big increase), A_6 = (too big increase). Then, define fuzzy sets $A_1, A_2, \ldots, A_6$ on the universe of discourse U as follows:

$$\begin{aligned}
A_1 &= 1/u_1 + 0.5/u_2 + 0/u_3 + 0/u_4 + 0/u_5 + 0/u_6,\\
A_2 &= 0.5/u_1 + 1/u_2 + 0.5/u_3 + 0/u_4 + 0/u_5 + 0/u_6,\\
A_3 &= 0/u_1 + 0.5/u_2 + 1/u_3 + 0.5/u_4 + 0/u_5 + 0/u_6,\\
A_4 &= 0/u_1 + 0/u_2 + 0.5/u_3 + 1/u_4 + 0.5/u_5 + 0/u_6,\\
A_5 &= 0/u_1 + 0/u_2 + 0/u_3 + 0.5/u_4 + 1/u_5 + 0.5/u_6,\\
A_6 &= 0/u_1 + 0/u_2 + 0/u_3 + 0/u_4 + 0.5/u_5 + 1/u_6.
\end{aligned} \tag{12}$$

Step 4: Fuzzify the values of historical data. If the number of variation of the enrollment of year i is p, where $p \in u_j$, and if there is a value represented by fuzzy set A_k in which the maximum membership value occurs at u_j, then p is translated to A_k. The fuzzified variations of the enrollment data are shown in Table 2.

Table 2
Fuzzified historical enrollments

Years	Variations	Fuzzified variations
1971		
1972	+ 508	A_4
1973	+ 304	A_4
1974	+ 829	A_5
1975	+ 764	A_5
1976	− 149	A_3
1977	+ 292	A_4
1978	+ 258	A_4
1979	+ 946	A_5
1980	+ 112	A_3
1981	− 531	A_2
1982	− 955	A_1
1983	+ 64	A_3
1984	− 352	A_2
1985	+ 18	A_3
1986	+ 821	A_5
1987	+ 875	A_5
1988	+ 1291	A_6
1989	+ 820	A_5
1990	+ 358	A_4
1991	+ 9	A_3
1992	− 461	A_2

Step 5: Choose a suitable window basis w, and calculate the output from the operation matrix $O^w(t)$ and the criterion matrix $C(t)$, where t is the year for which we want to forecast the enrollment. For example, if we set $w = 5$, then we can set a 4×6 operation matrix $O^5(t)$ and a 1×6 criterion matrix $C(t)$. Because $w = 5$, we must use six past years enrollment data, so we begin to forecast in 1977. In this case, the operation matrix $O^5(t)$ and the criterion matrix $C(t)$ are as follows:

$$O^5(1977) = \begin{bmatrix} \text{fuzzy variation of the enrollment of 1975} \\ \text{fuzzy variation of the enrollment of 1974} \\ \text{fuzzy variation of the enrollment of 1973} \\ \text{fuzzy variation of the enrollment of 1972} \end{bmatrix} = \begin{bmatrix} A_5 \\ A_5 \\ A_4 \\ A_4 \end{bmatrix}$$

$$= \begin{bmatrix} \text{(big decrease)} & \text{(decrease)} & \text{(no change)} & \text{(increase)} & \text{(big increase)} & \text{(too big increase)} \\ 0 & 0 & 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 & 0.5 & 0 \\ 0 & 0 & 0.5 & 1 & 0.5 & 0 \end{bmatrix},$$

$C(1977) =$ fuzzy variation of the enrollment of $1976 = [A_3]$

$$= [\begin{matrix} \text{(big decrease)} & \text{(decrease)} & \text{(no change)} & \text{(increase)} & \text{(big increase)} & \text{(too big increase)} \\ 0 & 0.5 & 1 & 0.5 & 0 & 0 \end{matrix}].$$

Calculate the relation matrix $R(t)$ by $R(t)[i,j] = O^w(t)[i,j] \times C(t)[j]$, where $1 \leqslant i \leqslant 4$, and $1 \leqslant j \leqslant 6$. Then, based on formula (10), we can get

$$R(1997) = \begin{bmatrix} \text{(big decrease)} & \text{(decrease)} & \text{(no change)} & \text{(increase)} & \text{(big increase)} & \text{(too big increase)} \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}.$$

Based on formula (11), we can get the fuzzified forecasting variation $F(1977)$ of year 1977 shown as follows:

$$F(1977) = [\begin{matrix} \text{(big decrease)} & \text{(decrease)} & \text{(no change)} & \text{(increase)} & \text{(big increase)} & \text{(too big increase)} \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \end{matrix}].$$

The fuzzified forecasted variations for the remaining years can be calculated by the same way and all the results are listed in Table 3.

Step 6: Defuzzify the fuzzy forecasted variations derived in Step 5. In this paper, we use the following principles to defuzzify the fuzzified forecasted variations:

(1) If the grades of membership of the fuzzified forecasted variation have only one maximum u_i, and the midpoint of u_i is m_i, then the forecasted variation is m_i. If the grades of membership of the fuzzified forecasted variation have more than one maximum $u_1, u_2, \ldots, u_k$, and their midpoints are $m_1, m_2, \ldots, m_k$, respectively, then the forecasted variation is $(m_1 + m_2 + \cdots + m_k)/k$. For example, from Table 3, we can see that the maximum membership value of $F(1977)$ is 0.5 which occurs at u_3 and u_4, where the midpoint of u_3 is 0 and the midpoint of u_4 is 400. The forecasted variation of 1977 is $(0 + 400)/2 = 200$.

(2) If the grades of membership of the fuzzified forecasted variation are all 0, then we set the forecasted variation to 0.

Step 7: Calculate the forecasted enrollments. The forecasted enrollment is forecasted variation plus the number of actual enrollment of last year. For example, if the forecasted variation in 1977 is 200, and the

Table 3
Forecasted variations with the window basis $w = 5$

Years	Membership functions of forecasted variations					
	u_1	u_2	u_3	u_4	u_5	u_6
1977	0	0	0.5	0.5	0	0
1978	0	0	0.5	1	0.5	0
1979	0	0	0.5	1	0.5	0
1980	0	0	0	0.5	1	0.25
1981	0	0.25	1	0.5	0	0
1982	0	0.5	0.5	0	0	0
1983	0.5	0.5	0	0	0	0
1984	0	0.5	1	0.25	0	0
1985	0.5	1	0.5	0	0	0
1986	0	0.5	1	0.25	0	0
1987	0	0	0	0.25	0	0
1988	0	0	0	0.25	1	0.25
1989	0	0	0	0	0.5	0.5
1990	0	0	0	0.25	1	0.5
1991	0	0	0	0.5	0.5	0
1992	0	0	0.5	0.5	0	0

Table 4
Forecasted results using the proposed fuzzy time-series method with the window basis $w = 5$

Years	Actual enrollments	Forecasted enrollments	Errors
1977	15603	15511	0.59%
1978	15861	16003	0.90%
1979	16807	16261	3.25%
1980	16919	17607	4.07%
1981	16388	16919	3.24%
1982	15433	16188	4.89%
1983	15497	14833	4.28%
1984	15145	15497	2.32%
1985	15163	14745	2.76%
1986	15984	15163	5.14%
1987	16859	16384	2.82%
1988	18150	17659	2.71%
1989	18970	19150	0.95%
1990	19328	19770	2.29%
1991	19337	19928	3.06%
1992	18876	19537	3.50%

actual enrollment in 1976 is 15311, then the forecasted enrollment of 1977 is 15311 + 200 = 15511. The results of the forecasted enrollment of the University of Alabama are shown in Table 4. The following error of each year by the proposed method under the window basis $w = 5$ is also shown in Table 4. The curve of the actual enrollments and the forecasted enrollments are shown in Fig. 1, where the window basis is 5.

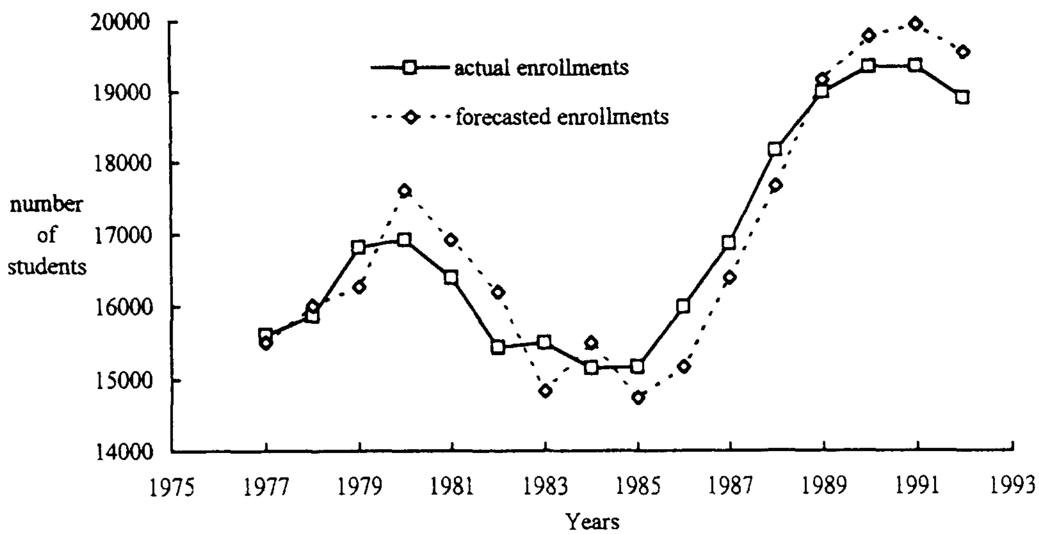

Fig. 1. The curve of the actual enrollments and forecasted enrollments.

4. Performance analysis

In Section 3, we have used the following method to forecast the enrollments of the University of Alabama:

$$R(t) = O^{w}(t) \otimes C(t) = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ R_{w1} & R_{w2} & \cdots & R_{wm} \end{bmatrix},$$

$$F(t) = [\mathrm{Max}(R_{11}, R_{21}, \ldots, R_{w1})\ \mathrm{Max}(R_{12}, R_{22}, \ldots, R_{w2}) \ldots \mathrm{Max}(R_{1m}, R_{2m}, \ldots, R_{wm})]$$

$$= [Q_1\ Q_2\ \ldots\ Q_m],$$

where $F(t)$ is the forecasted variation of year t represented by a fuzzy set, $O^{w}(t)$ is the operation matrix, $R(t)$ is a fuzzy relation which indicates the fuzzy relationship between the enrollments of last year and the other past years in data variations and "×" is the multiplication operation. The time complexity for calculating the fuzzy relation $R(t)$ is $O(n)$, and the time complexity for calculating the forecasted enrollment $F(t)$ is $O(wn)$, where w is the window basis and n is the number of elements in the universe of discourse. In [7, 8], Song et al. used the formula $A_i = A_{i-1} \circ R$ to forecast the enrollments of the University of Alabama, where A_{i-1} is the enrollment of year $i-1$ represented by a fuzzy set, A_i is the forecasted enrollment of year i represented by a fuzzy set, R is the union of fuzzy relations, and "$\circ$" is the Max–Min composition operator. It must take the time complexity of $O(kn^2)$ to calculate the Max–Min composition operations, where k is the number of fuzzy logical relationships and n is the number of elements in the universe of discourse. Because $O(kn^2) \gg O(wn)$, the proposed method is more efficient than the ones presented in [7, 8].

Table 5 shows the forecasting results of different window bases w ranging from 2 to 9. From Table 6 we can see that the average forecasting errors for different window bases range from 3.12% down to 2.79%, which are better than 3.18% in [7] and better than the one shown in [8] which ranges from 4.49% down to 3.15%.

Table 5
Forecasting enrollments with different window bases

Years	Actual enrollments	Forecasted enrollments							
		Window bases							
		$w = 2$	$w = 3$	$w = 4$	$w = 5$	$w = 6$	$w = 7$	$w = 8$	$w = 9$
1974	14696	14267							
1975	15460	15296	15296						
1976	15311	16260	16260	16260					
1977	15603	15711	15711	15511	15511				
1978	15861	15803	16003	16003	16003	16003			
1979	16807	16261	16261	16261	16261	16261	16261		
1980	16919	17407	17407	17407	17607	17607	17607	17607	
1981	16388	17319	17119	17119	16919	16919	16919	16919	16919
1982	15433	16188	16188	16188	16188	16188	16188	16188	16188
1983	15497	14833	14833	14833	14833	14833	14833	14833	14833
1984	15145	15097	15297	15497	15497	15497	15497	15497	15497
1985	15163	14945	14745	14745	14745	14745	14745	14745	14745
1986	15984	14963	15163	15163	15163	15163	15163	15163	15163
1987	16859	16384	16384	16384	16384	16384	16384	16784	16784
1988	18150	17659	17659	17659	17659	17659	17659	17659	17659
1989	18970	19150	19150	19150	19150	19150	19150	19150	19150
1990	19328	19970	19770	19770	19770	19770	19770	19770	19770
1991	19337	19928	19928	19928	19928	19728	19728	19728	19728
1992	18876	19537	19537	19537	19537	19537	19337	19337	19337

Table 6
Forecasting errors with different window bases

	Window bases							
	$w = 2$	$w = 3$	$w = 4$	$w = 5$	$w = 6$	$w = 7$	$w = 8$	$w = 9$
Average forecasting errors	2.99%	2.94%	3.12%	2.92%	3.01%	3.08%	2.89%	2.79%

It shows that the proposed method gets better forecasting results than the ones presented in [7, 8]. In Table 7, we compare the forecasting enrollments of the University of Alabama under different forecasting methods. In Table 8, we compare the average forecasting errors of different forecasting methods with the proposed method under window basis $w = 4$. The definition of the window basis w in the proposed method is like the definition of the model basis in [8], and they all use the previous w years of data to forecast the enrollment of this year, where the difference is that the proposed method uses actual variation to forecast, and the method in [8] uses actual enrollments to forecast. In Table 9, we compare the actual forecasting errors of these two methods. From Table 8, we can see that the proposed method is more efficient than the ones presented in [7, 8, 10] due to the fact that the proposed method simplifies the arithmetic operation process. Furthermore, we can also see that the average forecasting error of the proposed method is smaller than the ones presented in [2, 7, 8].

Table 7
The forecasting results of different forecasting methods

Years	Actual enrollments	Song–Chissom method [7]	Song–Chissom method [8] (under model basis $w = 4$ and using neural net method)	Chen's method [2]	Markov method [10] (time-invariant)	The proposed method (under window basis $w = 4$)
1972	13 563	14 000		14 000	13 563	
1973	13 867	14 000		14 000	13 867	
1974	14 696	14 000		14 000	14 696	
1975	15 460	15 500	14 700	15 500	15 460	
1976	15 311	16 000	14 800	16 000	15 311	16 260
1977	15 603	16 000	15 400	16 000	15 603	15 511
1978	15 861	16 000	15 500	16 000	15 861	16 003
1979	16 807	16 000	15 500	16 000	16 807	16 261
1980	16 919	16 813	16 800	16 833	16 919	17 407
1981	16 388	16 813	16 200	16 833	16 388	17 119
1982	15 433	16 789	16 400	16 833	15 433	16 188
1983	15 497	16 000	16 800	16 000	15 497	14 833
1984	15 145	16 000	16 400	16 000	15 145	15 497
1985	15 163	16 000	15 500	16 000	15 163	14 745
1986	15 984	16 000	15 500	16 000	15 984	15 163
1987	16 859	16 000	15 500	16 000	16 859	16 384
1988	18 150	16 813	16 800	16 833	18 150	17 659
1989	18 970	19 000	19 300	19 000	18 970	19 150
1990	19 328	19 000	17 800	19 000	19 328	19 770
1991	19 337	19 000	19 300	19 000	19 337	19 928
1992	18 876	not forecasted	19 600	19 000	not forecasted	19 537

Table 8
A comparison of the average forecasting errors of different forecasting methods (Note: k denotes the number of fuzzy logical relationships, n denotes the number of elements in the universe of discourse, p denotes the number of fuzzy logical relationship groups, c denotes the number of transitions in the historical data, and w denotes the window basis)

	Song–Chissom method [7]	Song–Chissom method [8] (under model basis $w = 4$ and using neural net method)	Chen's method [2]	Markov method [10] (time-invariant)	The proposed method (under window basis ($w = 4$)
Style	Time-invariant	Time-variant	Time-invariant	Time-invariant	Time-variant
Time complexity	$O(kn^2)$	$O(kn^2)$	$O(p)$	$O(cn^2)$	$O(wn)$
Average forecasting errors	3.2%	4.37%	3.22%	2.6%	3.12%

5. Conclusions

In this paper, we have proposed a new method for handling forecasting problems based on fuzzy time series, where the data of historical enrollments of the University of Alabama shown in [7, 8] are adopted to illustrate the forecasting process. We have also shown that the proposed method is more efficient than the ones presented in [7, 8, 10]. Furthermore, the average forecasting error of the proposed method is smaller than the ones presented in [2, 7, 8]. From Section 3, we can see that the bigger the window basis w, the more

Table 9
A comparison of average forecasting errors of the proposed method and Song–Chissom method

	Window bases (or Model bases)							
	$w = 2$	$w = 3$	$w = 4$	$w = 5$	$w = 6$	$w = 7$	$w = 8$	$w = 9$
Forecasting errors of the Song–Chissom method [8] (using neural net method)	3.15%	3.89%	4.37%	4.41%	4.49%	4.35%	4.45%	4.23%
Forecasting errors of the proposed method	2.99%	2.94%	3.12%	2.92%	3.01%	3.08%	2.89%	2.79%

time is needed to perform calculations. From Table 6, we can see that the biggest average forecasting error occurred at $w = 4$, and the smallest forecasting error occurred at $w = 9$. It is difficult to find the relationships between the window basis and the average forecasting error, but there is an efficient way [3] which uses genetic algorithms to find the better window basis to forecast the enrollments. From Table 8, we can see that the proposed method has a better average forecasting accuracy than the other methods presented in [2, 7, 8]. Furthermore, from Table 8, we can also see that the time complexity of the proposed method is better than the ones presented in [7, 8, 10].

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