

Bayesian Networks: Independencies and Inference

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What Independencies does a Bayes Net Model?

- In order for a Bayesian network to model a probability distribution, the following must be true by definition:

Each variable is conditionally independent of all its non-descendants in the graph given the value of all its parents.

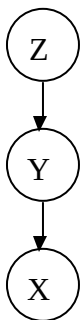
- This implies

$$P(X_1 \dots X_n) = \prod_{i=1}^n P(X_i \mid \text{parents}(X_i))$$

- But what else does it imply?

What Independencies does a Bayes Net Model?

- Example:



Given Y , does learning the value of Z tell us nothing new about X ?

I.e., is $P(X|Y, Z)$ equal to $P(X | Y)$?

Yes. Since we know the value of all of X 's parents (namely, Y), and Z is not a descendant of X , X is conditionally independent of Z .

Also, since independence is symmetric,
 $P(Z|Y, X) = P(Z|Y)$.

Quick proof that independence is symmetric

- Assume: $P(X|Y, Z) = P(X|Y)$
- Then:

$$P(Z | X, Y) = \frac{P(X, Y | Z)P(Z)}{P(X, Y)} \quad (\text{Bayes's Rule})$$

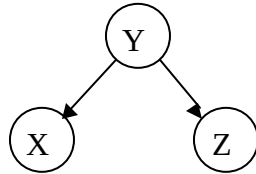
$$= \frac{P(Y | Z)P(X | Y, Z)P(Z)}{P(X | Y)P(Y)} \quad (\text{Chain Rule})$$

$$= \frac{P(Y | Z)P(X | Y)P(Z)}{P(X | Y)P(Y)} \quad (\text{By Assumption})$$

$$= \frac{P(Y | Z)P(Z)}{P(Y)} = P(Z | Y) \quad (\text{Bayes's Rule})$$

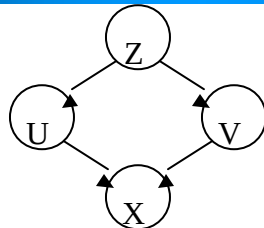
What Independencies does a Bayes Net Model?

- Let $I\langle X, Y, Z \rangle$ represent X and Z being conditionally independent given Y .



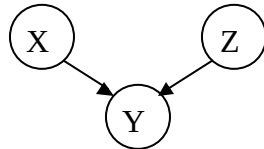
- $I\langle X, Y, Z \rangle$? Yes, just as in previous example: All X 's parents given, and Z is not a descendant.

What Independencies does a Bayes Net Model?



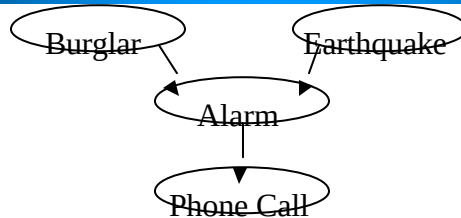
- $I\langle X, \{U\}, Z \rangle$? No.
- $I\langle X, \{U, V\}, Z \rangle$? Yes.
- Maybe $I\langle X, S, Z \rangle$ iff S acts a cutset between X and Z in an undirected version of the graph...?

Things get a little more confusing



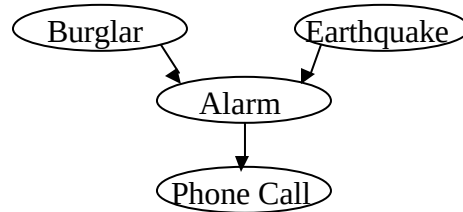
- X has no parents, so we know all its parents' values trivially
- Z is not a descendant of X
- So, $I\langle X, \{\}, Z \rangle$, even though there's an undirected path from X to Z through an unknown variable Y.
- What if we do know the value of Y, though? Or one of its descendants?

The “Burglar Alarm” example



- Your house has a twitchy burglar alarm that is also sometimes triggered by earthquakes.
- Earth arguably doesn't care whether your house is currently being burgled
- While you are on vacation, one of your neighbors calls and tells you your home's burglar alarm is ringing. Uh oh!

Things get a lot more confusing



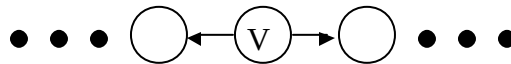
- But now suppose you learn that there was a medium-sized earthquake in your neighborhood. Oh, whew! Probably not a burglar after all.
- Earthquake “explains away” the hypothetical burglar.
- But then it must **not** be the case that
 $I\langle \text{Burglar}, \{\text{Phone Call}\}, \text{Earthquake} \rangle$, even though
 $I\langle \text{Burglar}, \{\}, \text{Earthquake} \rangle$!

d-separation to the rescue

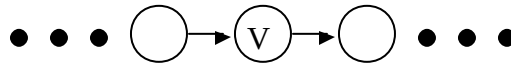
- Fortunately, there is a relatively simple algorithm for determining whether two variables in a Bayesian network are conditionally independent: *d-separation*.
- Definition: X and Z are *d-separated* by a set of evidence variables E iff every undirected path from X to Z is “blocked”, where a path is “blocked” iff one or more of the following conditions is true: ...

A path is “blocked” when...

- There exists a variable V on the path such that
 - it **is** in the evidence set E
 - the arcs putting V in the path are “tail-to-tail”



- Or, there exists a variable V on the path such that
 - it **is** in the evidence set E
 - the arcs putting V in the path are “tail-to-head”



- Or, ...

A path is “blocked” when... (the funky case)

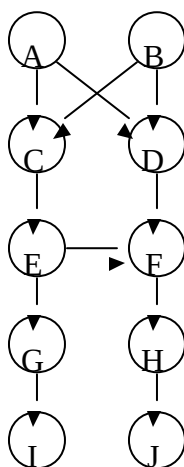
- ... Or, there exists a variable V on the path such that
 - it **is NOT** in the evidence set E
 - **neither are any of its descendants**
 - the arcs putting V on the path are “head-to-head”



d-separation to the rescue, cont'd

- Theorem [Verma & Pearl, 1998]:
 - If a set of evidence variables E d-separates X and Z in a Bayesian network's graph, then $I\langle X, E, Z \rangle$.
- d -separation can be computed in linear time using a depth-first-search-like algorithm.
- Great! We now have a fast algorithm for automatically inferring whether learning the value of one variable might give us any additional hints about some other variable, given what we already know.
 - “Might”: Variables may actually be independent when they're not d-separated, depending on the actual probabilities involved

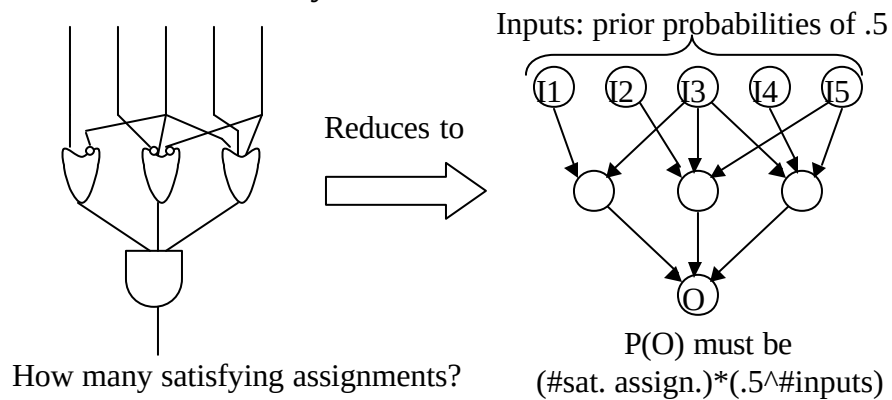
d-separation example



- $I\langle C, \{\}, D \rangle?$
- $I\langle C, \{A\}, D \rangle?$
- $I\langle C, \{A, B\}, D \rangle?$
- $I\langle C, \{A, B, J\}, D \rangle?$
- $I\langle C, \{A, B, E, J\}, D \rangle?$

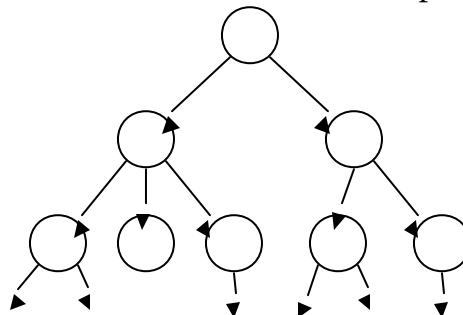
Bayesian Network Inference

- Inference: calculating $P(X|Y)$ for some variables or sets of variables X and Y .
- Inference in Bayesian networks is #P-hard!



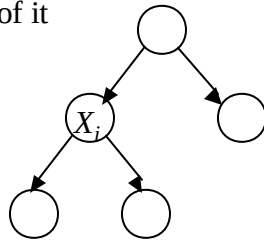
Bayesian Network Inference

- **But...** inference is still tractable in some cases.
- Let's look at a special class of networks: *trees* / *forests* in which each node has at most one parent.



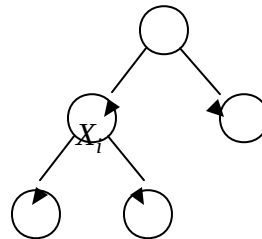
Decomposing the probabilities

- Suppose we want $P(X_i | E)$ where E is some set of evidence variables.
- Let's split E into two parts:
 - E_i^- is the part consisting of assignments to variables in the subtree rooted at X_i
 - E_i^+ is the rest of it



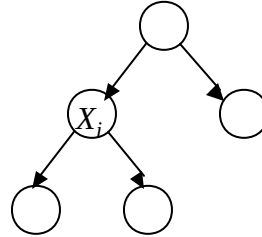
Decomposing the probabilities, cont'd

$$P(X_i | E) = P(X_i | E_i^-, E_i^+)$$



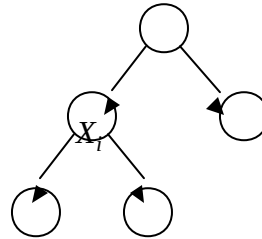
Decomposing the probabilities, cont'd

$$\begin{aligned}
 P(X_i | E) &= P(X_i | E_i^-, E_i^+) \\
 &= \frac{P(E_i^- | X, E_i^+) P(X | E_i^+)}{P(E_i^- | E_i^+)}
 \end{aligned}$$



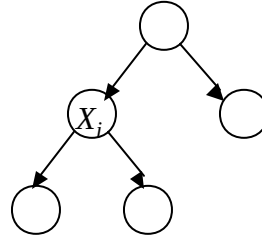
Decomposing the probabilities, cont'd

$$\begin{aligned}
 P(X_i | E) &= P(X_i | E_i^-, E_i^+) \\
 &= \frac{P(E_i^- | X, E_i^+) P(X | E_i^+)}{P(E_i^- | E_i^+)} \\
 &= \frac{P(E_i^- | X) P(X | E_i^+)}{P(E_i^- | E_i^+)}
 \end{aligned}$$



Decomposing the probabilities, cont'd

$$\begin{aligned}
 P(X_i | E) &= P(X_i | E_i^-, E_i^+) \\
 &= \frac{P(E_i^- | X, E_i^+) P(X | E_i^+)}{P(E_i^- | E_i^+)} \\
 &= \frac{P(E_i^- | X) P(X | E_i^+)}{P(E_i^- | E_i^+)} \\
 &= \alpha(X_i) \pi(X_i) \lambda(X_i)
 \end{aligned}$$



Where:

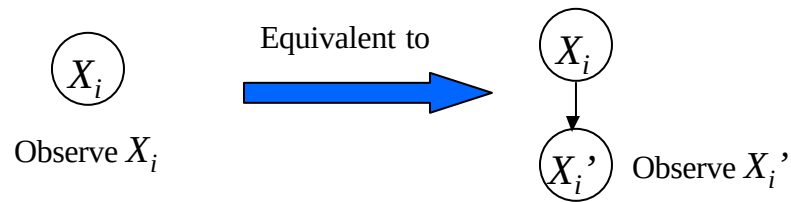
- α is a constant independent of X_i
- $\pi(X_i) = P(X_i | E_i^+)$
- $\lambda(X_i) = P(E_i^- | X_i)$

Using the decomposition for inference

- We can use this decomposition to do inference as follows. First, compute $\lambda(X_i) = P(E_i^- | X_i)$ for all X_i recursively, using the leaves of the tree as the base case.
- If X_i is a leaf:
 - If X_i is in E : $\lambda(X_i) = 1$ if X_i matches E , 0 otherwise
 - If X_i is not in E : E_i^- is the null set, so $P(E_i^- | X_i) = 1$ (constant)

Quick aside: “Virtual evidence”

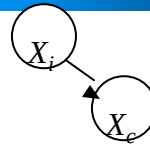
- For theoretical simplicity, but without loss of generality, let's assume that *all* variables in E (the evidence set) are leaves in the tree.
- Why can we do this WLOG:



Where $P(X_i' | X_i) = 1$ if $X_i' = X_i$, 0 otherwise

Calculating $\lambda(X_i)$ for non-leaves

- Suppose X_i has one child, X_c .

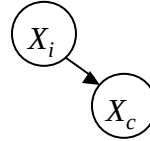


- Then:

$$\lambda(X_i) = P(E_i^- | X_i) =$$

Calculating $\lambda(X_i)$ for non-leaves

- Suppose X_i has one child, X_c .

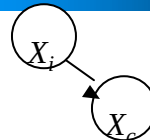


- Then:

$$\lambda(X_i) = P(E_i^- | X_i) = \sum_j P(E_i^-, X_c = j | X_i)$$

Calculating $\lambda(X_i)$ for non-leaves

- Suppose X_i has one child, X_c .

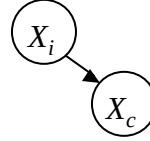


- Then:

$$\begin{aligned} \lambda(X_i) &= P(E_i^- | X_i) = \sum_j P(E_i^-, X_c = j | X_i) \\ &= \sum_j P(X_c = j | X_i) P(E_i^- | X_i, X_c = j) \end{aligned}$$

Calculating $\lambda(X_i)$ for non-leaves

- Suppose X_i has one child, X_c .



- Then:

$$\begin{aligned}
 \lambda(X_i) &= P(E_i^- | X_i) = \sum_j P(E_i^-, X_c = j | X_i) \\
 &= \sum_j P(X_c = j | X_i) P(E_i^- | X_i, X_c = j) \\
 &= \sum_j P(X_c = j | X_i) P(E_i^- | X_c = j) \\
 &= \sum_j P(X_c = j | X_i) \lambda(X_c)
 \end{aligned}$$

Calculating $\lambda(X_i)$ for non-leaves

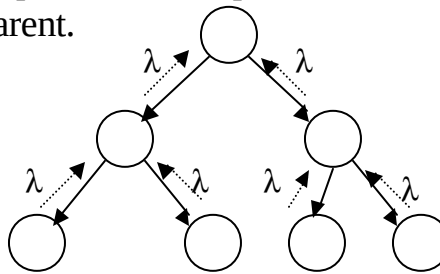
- Now, suppose X_i has a set of children, C .
- Since X_i *d-separates* each of its subtrees, the contribution of each subtree to $\lambda(X_i)$ is independent:

$$\begin{aligned}
 \lambda(X_i) &= P(E_i^- | X_i) = \prod_{X_j \in C} \lambda_j(X_i) \\
 &= \prod_{X_j \in C} \left[\sum_j P(X_j | X_i) \lambda_j(X_j) \right]
 \end{aligned}$$

where $\lambda_j(X_i)$ is the contribution to $P(E_i^- | X_i)$ of the part of the evidence lying in the subtree rooted at one of X_i 's children X_j .

We are now λ -happy

- So now we have a way to recursively compute all the $\lambda(X_i)$'s, starting from the root and using the leaves as the base case.
- If we want, we can think of each node in the network as an autonomous processor that passes a little “ λ message” to its parent.



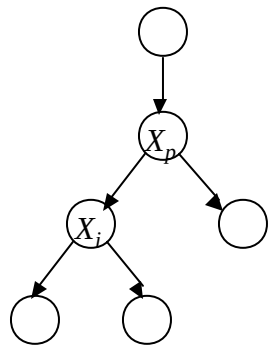
The other half of the problem

- Remember, $P(X_i|E) = \alpha\pi(X_i)\lambda(X_i)$. Now that we have all the $\lambda(X_i)$'s, what about the $\pi(X_i)$'s?

$$\pi(X_i) = P(X_i | E_i^+).$$

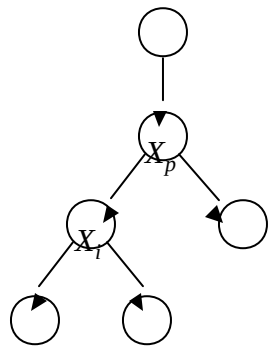
- What about the root of the tree, X_r ? In that case, E_r^+ is the null set, so $\pi(X_r) = P(X_r)$. No sweat. Since we also know $\lambda(X_r)$, we can compute the final $P(X_r)$.
- So for an arbitrary X_i with parent X_p , let's inductively assume we know $\pi(X_p)$ and/or $P(X_p|E)$. How do we get $\pi(X_i)$?

Computing $\pi(X_i)$



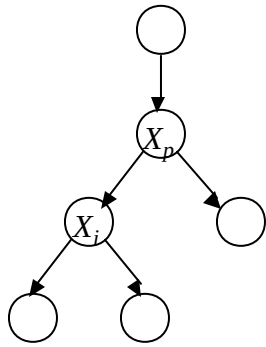
$$p(X_i) = P(X_i | E_i^+) =$$

Computing $\pi(X_i)$



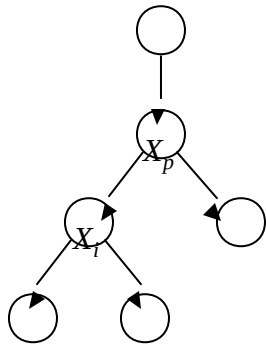
$$p(X_i) = P(X_i | E_i^+) = \sum_j P(X_i, X_p = j | E_i^+)$$

Computing $\pi(X_i)$



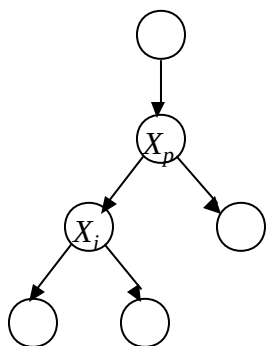
$$\begin{aligned} p(X_i) &= P(X_i | E_i^+) = \sum_j P(X_i, X_p = j | E_i^+) \\ &= \sum_j P(X_i | X_p = j, E_i^+) P(X_p = j | E_i^+) \end{aligned}$$

Computing $\pi(X_i)$



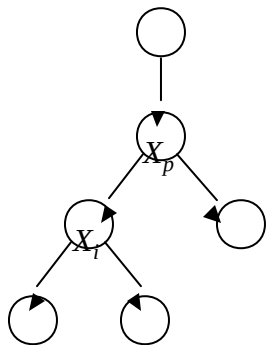
$$\begin{aligned} p(X_i) &= P(X_i | E_i^+) = \sum_j P(X_i, X_p = j | E_i^+) \\ &= \sum_j P(X_i | X_p = j, E_i^+) P(X_p = j | E_i^+) \\ &= \sum_j P(X_i | X_p = j) P(X_p = j | E_i^+) \end{aligned}$$

Computing $\pi(X_i)$



$$\begin{aligned}
 p(X_i) &= P(X_i | E_i^+) = \sum_j P(X_i, X_p = j | E_i^+) \\
 &= \sum_j P(X_i | X_p = j, E_i^+) P(X_p = j | E_i^+) \\
 &= \sum_j P(X_i | X_p = j) P(X_p = j | E_i^+) \\
 &= \sum_j P(X_i | X_p = j) \frac{P(X_p = j | E)}{\pi_i(X_p = j)}
 \end{aligned}$$

Computing $\pi(X_i)$

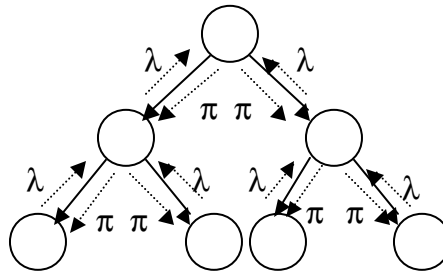


$$\begin{aligned}
 p(X_i) &= P(X_i | E_i^+) = \sum_j P(X_i, X_p = j | E_i^+) \\
 &= \sum_j P(X_i | X_p = j, E_i^+) P(X_p = j | E_i^+) \\
 &= \sum_j P(X_i | X_p = j) P(X_p = j | E_i^+) \\
 &= \sum_j P(X_i | X_p = j) \frac{P(X_p = j | E)}{\pi_i(X_p = j)} \\
 &= \sum_j P(X_i | X_p = j) \pi_i(X_p = j)
 \end{aligned}$$

Where $\pi_i(X_p)$ is defined as $\frac{P(X_p | E)}{\pi_i(X_p)}$

We're done. Yay!

- Thus we can compute all the $\pi(X_i)$'s, and, in turn, all the $P(X_i|E)$'s.
- Can think of nodes as autonomous processors passing λ and π messages to their neighbors

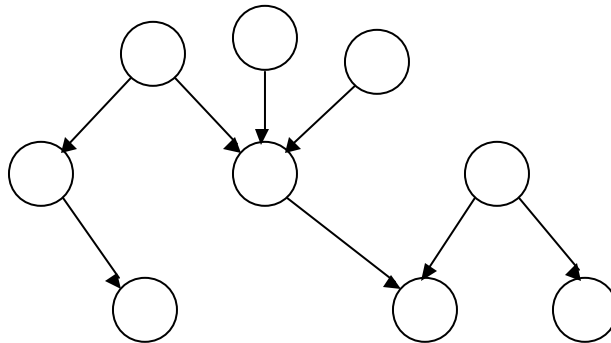


Conjunctive queries

- What if we want, e.g., $P(A, B | C)$ instead of just marginal distributions $P(A | C)$ and $P(B | C)$?
- Just use chain rule:
 - $P(A, B | C) = P(A | C) P(B | A, C)$
 - Each of the latter probabilities can be computed using the technique just discussed.

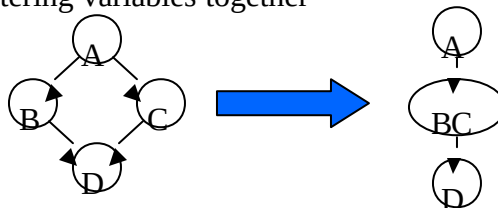
Polytrees

- Technique can be generalized to *polytrees*: undirected versions of the graphs are still trees, but nodes can have more than one parent

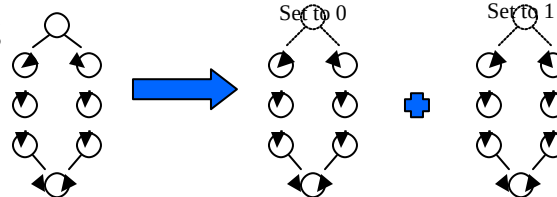


Dealing with cycles

- Can deal with undirected cycles in graph by
 - clustering variables together

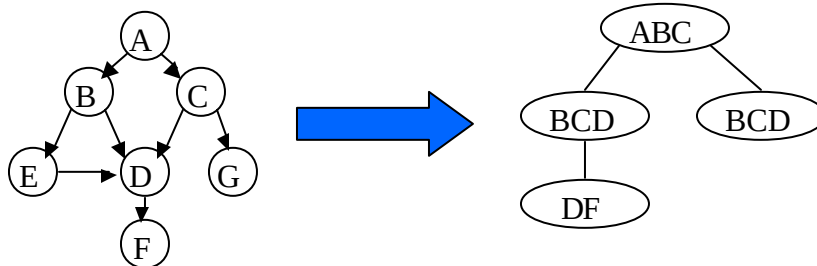


- Conditioning



Join trees

- Arbitrary Bayesian network can be transformed via some evil graph-theoretic magic into a *join tree* in which a similar method can be employed.



In the worst case the join tree nodes must take on exponentially many combinations of values, but often works well in practice