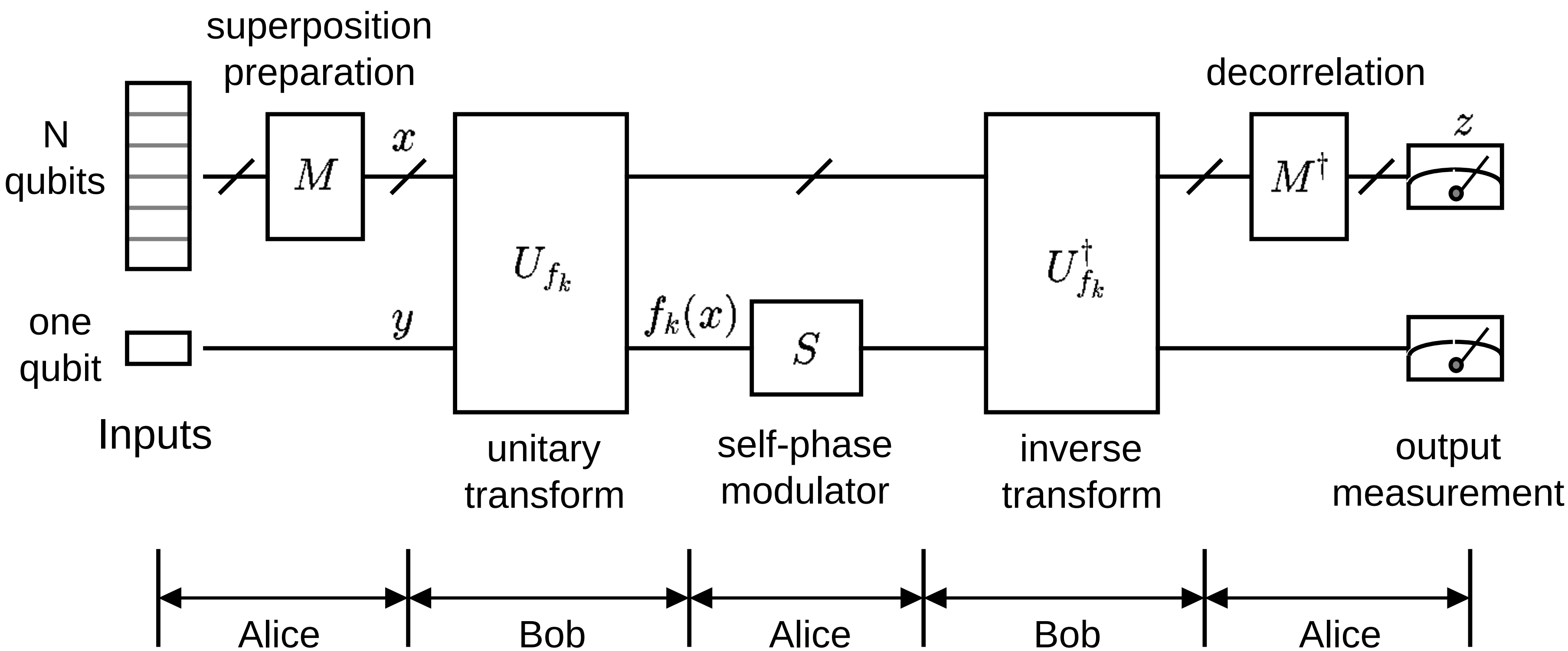

Figure 1

I. Chuang PRA qcomp-28mar95

A Simple Quantum Computer

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We propose an implementation of a quantum computer to solve Deutsch's problem, which requires exponential time on a classical computer but only linear time with quantum parallelism. By using a dual-rail qubit representation as a simple form of error correction, our machine can tolerate some amount of decoherence and still give the correct result with high probability. The design which we employ also demonstrates a signature for quantum parallelism which unambiguously deliniates the desired quantum behavior from the merely classical. The experimental demonstration of our proposal using quantum optical components calls for the development of several key technologies common to single photonics.

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I. INTRODUCTION

The field of quantum computation has received tremendous new interest since the recent result of Shor [1], which shows the possibility of using the non-local behavior of quantum mechanics to factor integers in random polynomial time. This is exponentially better than is achievable on a comparable classical machine, with any algorithm known today.

However, there is a catch. Quantum computing (like quantum cryptography) relies fundamentally on the processing of bits of information which can be superpositions of logical one and zero. As long as the mutual coherence among a set of quantum bits (qubits) [2] is preserved, they can simultaneously take on more than one value, giving rise to a useful effect known as quantum parallelism. With sufficient cleverness, algorithms can be devised which take advantage of this effect to solve some problems faster than is possible with a classical computer.

The catch is that these qubits are "Schrödinger cat" states, which are normally highly susceptible to collapse. Whenever a qubit is observed by an external agent (such as the environment [3]), coherence with other qubits in the system is partially lost due to the collapse of its wavefunction. This loss of coherence is accompanied by a loss of information [4] which is likely to cause a malfunction of the quantum computer. Thus, simply put, the practicality of using quantum parallelism is crucially dependent on our ability to build a machine which is sufficiently perfect and isolated from its environment so as to preserve quantum coherence throughout a calculation [6].

The key question upon which the feasibility of quantum computing hinges is how difficult it is to maintain quantum coherence in a real implementation. This is very much a *system* issue, because to succeed, not only must the logic devices be perfect, but also, the scheme for their interconnection, and the method for preparing and extracting the inputs and outputs of the computer. Although implementations of several quantum-mechanical logic gates [5,7] and general architectures [8,9] have been proposed, no designs for a specific machine have yet appeared in the literature, and therefore, it is unclear what the minimum requirement is for realizing a complete system. As a result, it is also difficult to pin down what noise issues limit the feasibility of maintaining quantum coherence in a complete quantum computer.

The purpose of this article is to remedy this problem by proposing a specific realization of a quantum computer which solves Deutsch's problem [10]. Although the machine which we envision has little practical use, it is a simple system which (1) demonstrates the concept of quantum parallelism, and (2) delineates the desired quantum behavior from the merely classical by using simple error correction. The approach which we outline also describes several techniques which we believe will be useful in constructing a more general purpose machine.

We note in relation to the literature that many issues which arise in the course of our discussion remain open questions. In particular, we do not attempt to address the problem of synthesizing a *universal* quantum computer from some minimal set of logic gates [11,9,12]. Neither are we particularly interested in solving the full problem of quantum error-correction [13,14]. Instead, our concern is the *reality* of quantum computing. By focusing on the complete design of a specific machine, we learn about realizability, operation, and robustness – system issues which are of principle concern in understanding the impact of decoherence. Our design of a simple quantum computer using error correction provides a concrete and new framework for analyzing the role of decoherence in quantum computing.

We begin by summarizing Deutsch's problem. We then compare the classical and quantum solutions to a simplified version of the problem, and discuss how the required components may be realized. This leads us to a design for a machine which we present in Section 4, which is followed by an analysis of its error correcting ability in Section 5. We conclude with a discussion of the experimental possibilities.

II. DEUTSCH'S PROBLEM

Deutsch's problem may be described as the following game. Alice, in Amsterdam, selects a number x from 0 to $2L-1$, and mails it in a letter to Bob, in Boston. Bob calculates some function $f(x)$ and replies with the result, which is either 0 or 1. Now, Bob has agreed to use one of only two kinds of functions, either type (1), which are constant for all values of x, or type (2), which are equal to one for exactly half of all the possible x. Alice's mission is to determine with certainty which type of function Bob has chosen by corresponding with him the fewest number of times. How fast can she succeed?

In the classical case, Alice may only send Bob one value of x in each letter. At worst, Alice will need to query Bob at least $L+1$ times, since she may receive, e.g., L zeros before finally getting a one, telling her that Bob's function is type 2. The best deterministic classical algorithm she can use therefore requires $L+1$ queries. Note that in each letter, Alice sends Bob N bits of information, where $N = \log_2(2L)$.

Now add a new twist to the problem. Suppose that Bob and Alice can exchange quantum bits (instead of just classical bits), and furthermore, Bob calculates $f(x)$ using a unitary transformation U_f. Alice can now get back more than one value of $f(x)$ from Bob in a single query, while still exchanging only about N bits. For example, Alice may send Bob an atom trap containing $N+1$ two-level atoms. The first N atoms, representing x, are prepared in an equal superposition of their excited and ground states, while the last atom, a scratch-pad for the result $y = f(x)$, is put in its ground state. In Boston, Bob uses a sequence of electromagnetic pulses to unitarily put atom y in the state $f(x)$. Note that x is in a superposition of all values $[0, 2^N - 1]$, and therefore, y is left a superposition of all possible values of $f(x)$. However, when Alice receives the reply, she can't achieve her mission simply by measuring atom y, since that would collapse the superposition state and give her only one result!

Instead, Alice must be more clever. She gives y a π phase shift relative to x, then sends the qubits once more to Bob. This time, Bob agrees to calculate $U_f^\dagger$ instead of U_f, i.e., he inverts what he did before, leaving y in its ground state. Since y and x are entangled, this procedure also leaves the N qubits of x with a special relative phase, such that those values of x for which $f(x)$ is even are be $180°$ out of phase with the others. When Alice receives the result back from Bob, she can perform an interference experiment to determine the type of Bob's function, with certainty. This is accomplished using only *two* queries.

The quantum algorithm followed by Alice in the latter case was devised by Deutsch and Jozsa, and a more mathematical description can be found in their article [10]. A schematic of the algorithm is shown in Figure 1. This drawing, and our description above highlight the two principle differences between classical and quantum computing: (1) information is represented as quantum bits, and (2) information interactions are performed using unitary transformations. These two changes allow Deutsch's problem to be solved in $\mathcal{O}(N)$, rather than in $\mathcal{O}(\exp N)$ time. In our example, physical distance was used to artificially elevate the cost of calculating $f(x)$; this is not needed in general, where $f(x)$ may be inherently difficult to calculate. We shall study next how qubits can be generated, manipulated, interacted, and measured.

III. COMPONENTS OF A QUANTUM COMPUTER

The nature of the physical realization of the algorithm of Figure 1 depends most on the representation chosen for the quantum bit. As we mentioned, two-level atoms are one possibility. Single electrons, solitons, magnetic flux quanta, nuclear spins, and quantum dots are other possibilities which have been considered. We have chosen to represent qubits as single photons, primarily because almost all the required components (for a single photon quantum computer) exist today, but also because quantum optics is a well-developed field in which noise is a thoroughly understood subject. However, we believe that there are some general limitations governing all qubit representations, and our goal is to try to elucidate those, so despite our use of quantum optics terminology, it should be kept in mind that many of our conclusions are applicable to other systems as well.

Given that we are using $|0\rangle$ (the vacuum state) and $|1\rangle$ (the single photon state) to represent logical zero and one, respectively, we must answer the following three questions to construct our quantum computer to solve Deutsch's problem:

(1) How is a superposition state prepared?

(2) What unitary transform is used to calculate $f(x)$?

(3) What interference experiment is performed to determine the final result?

That is, we need devices to perform the unitary operations M, U_f, and S, and an architecture which provides a definite phase reference so as to allow the final interference experiment to be performed. We now show how the traditional tools of optics can be used to fulfill our needs. We shall use beamsplitters, mirrors, phase shifters, and Kerr media.

The first task is to create a superposition state. It is possible in principle to create the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big[|0\rangle + |1\rangle\Big] \tag{3.1}$$

but we have a simpler alternative. The ordinary 50/50 optical beamsplitter [15,16] acting on modes a and b is described by the quantum operator B, shown in Figure 2. Let us label states as $|\mathsf{ab}\rangle$. A beamsplitter with input $|01\rangle$ gives the output

$$B|01\rangle = \frac{1}{\sqrt{2}}\Big[|01\rangle + |10\rangle\Big] . \tag{3.2}$$

Now comes our first trick. Let us represent a single qubit by a pair of modes, such that $|01\rangle$ and $|10\rangle$ are logical zero and one, respectively. This *dual-rail* representation of a logical state embeds an elementary form of error correction which will be useful later. With this representation, we see that a simple beamsplitter can be used to generate the desired superposition state of logical zero and one.

Next, we must calculate $f(x)$ using a unitary transform. Since $f(x)$ is a mapping from $\mathcal{Z} \to \mathcal{Z}_2$, we may consider it to be calculable by an acyclic boolean circuit. It is therefore possible to implement it using a cascade of reversible logic gates, such as the Fredkin gate [17]. For example, consider the two-bit Deutsch problem. Here, $0 \leq x < 4$, and there exist eight possible functions which Bob may choose (Table I). Two circuits which can be used to implement $f(x)$ are shown in Figure 3B. Also shown are circuits for the one-bit problem, where $0 \leq x < 2$ (Table II). The reversible logic circuits correspond directly to unitary operators which may be implemented as quantum-mechanical transforms. This is done simply by using a quantum Fredkin gate in place of the classical one.

Note that this technique, of utilizing a reversible logic implementation to determine the unitary operator necessary to implement a classical function, is valid in the general case. For example, Shor's algorithm requires the calculation of $x^a \bmod N$, for which the proper unitary transform may be arrived at through analysis of the required reversible logic circuit. Also note that we have chosen the Fredkin gate in favor of the Toffoli gate, because *conservative* invertible logic gates conserve the number of "ones" and therefore are possibly more amicable to qubit representations where a logical one implies existence of some energy packet (as will be the case for our system) [18].

An optical realization of the quantum Fredkin gate (Figure 2) has been proposed [5], and is understood well. It is simply a nonlinear Mach-Zehnder interferometer, with an external control signal which causes the exchange of a and b by inducing a relative π phase shift in one arm via cross-phase modulation in the Kerr medium. This device may be viewed as a "controlled beamsplitter," where the c-input determines the angle of a beamsplitter with inputs a and b. We shall let $\chi = \pi$, such that when $c^\dagger c = 1$, the Fredkin operator F acts on a and b just like a beamsplitter with angle $\pi/2$, i.e., $F|101\rangle = -|011\rangle$ and $F|011\rangle = |101\rangle$, where the state is $|\mathsf{abc}\rangle$. Note that when $c^\dagger c = 0$, the Fredkin operator is the identity, $F = I$.

Note that each of the components of our quantum computer, which operate on *dual-rail* qubit representations, have a corresponding description in the traditional picture of *single-rail* qubit functions. A two-input beamsplitter operating on modes a and $\bar{\mathsf{a}}$ is equivalent to Deutsch's one-input $\sqrt{\text{NOT}}$ gate [19] acting on the qubit represented by the pair $\{\mathsf{a}, \bar{\mathsf{a}}\}$. Similarly, three three-input Fredkin gates acting on modes a, $\bar{\mathsf{a}}$, b, $\bar{\mathsf{b}}$, c, and $\bar{\mathsf{c}}$ can perform any thee-input Toffoli gate transform on the three quibits represented by the pairs $\{\mathsf{a}, \bar{\mathsf{a}}\}$, $\{\mathsf{b}, \bar{\mathsf{b}}\}$, and $\{\mathsf{c}, \bar{\mathsf{c}}\}$ [20]; in this sense, the Fredkin gate is close to DiVincenzo's "controlled-rotation" gate [9]. Incidentally, since it has been shown that these traditional gates are "universal," in the sense that they can be cascaded to synthesize any arbitrary quantum computing device, it follows that our component set is also universal.

One more unitary operator which is needed is the phase shift S performed by Alice after receiving the first letter back from Bob. This is accomplished using a π phase delay. Finally, the task of interference and measurement can be performed by using an interferometer and ideal photon counters. Alice can create and decorrelate superpositions using beamsplitters and communicate to Bob by sending him photons; and Bob can calculate his function using Fredkin gates. Thus, the Deutsch-Jozsa quantum algorithm may be implemented using the traditional components of quantum optics. This viewpoint will be useful in analyzing the physics of our machine as we assemble it in the following section.

IV. THE MACHINE

The one-bit Deutsch problem is the simple case where Alice sends Bob a value of $x = 0$ or $x = 1$, and Bob replies with $f(x)$, where he has chosen one of the four functions shown in Table II. Clearly, in the classical case, Alice can achieve her goal of determining the type of Bob's function by sending Bob just two queries. The quantum solution can be achieved with the same number of queries, so there is no time advantage in this case. However, it is worthwhile to consider precisely how the quantum algorithm is implemented to understand the role which quantum coherence plays.

The machine which we propose is diagrammed in Figure 4. The general operation is as follows. Alice prepares two qubits $\{\mathsf{a}, \mathsf{b}\}$ and $\{\mathsf{c}, \mathsf{d}\}$, each of which is represented by a dual-rail single-photon eigenstate. Operationally, this means that she sends single photon eigenstates simultaneously into modes d and b, and the vacuum state into the other two. The $\{\mathsf{c}, \mathsf{d}\}$ qubit is passed through a $\sqrt{\text{NOT}}$ gate which implemented by a beamsplitter to prepare a value of x which is in a 50/50 superposition of 0 and 1. This qubit is passed along with the scratch-pad qubit $\{\mathsf{a}, \mathsf{b}\}$ to Bob. Bob uses a quantum Fredkin gate and three classical switches to perform his calculation, and returns $f(x)$ in the scratch-pad. Alice gives the result a relative π phase shift, then allows Bob to invert his first transform. Finally, Alice sends the $\{\mathsf{c},\mathsf{d}\}$ qubit through a final beamsplitter, and measures the number of photons she receives in all four modes. In the absence of error, the detector for mode d tells Alice the type of Bob's function with certainty, from a single execution of the machine.

Let us now analyze the behavior of this machine by calculating the states $|\psi_i\rangle$, defined as

$$\begin{aligned}
|\psi_0\rangle &= \text{Alice's initial state}\\
|\psi_1\rangle &= \text{Superposition state sent to Bob}\\
|\psi_2\rangle &= \text{Result returned to Alice the first time}\\
|\psi_3\rangle &= \text{Phase shifted state sent back to Bob}\\
|\psi_4\rangle &= \text{Result returned to Alice the second time}\\
|\psi_5\rangle &= \text{Alice's final state, after decorrelation}\,.
\end{aligned}$$

We shall label the states as $|\mathsf{abcd}\rangle$, and use the fact that S acts on mode a, B acts on c and d, and U_f acts on a, b, and c. We may think of mode c of state $|\psi_1\rangle$ as the value of x prepared by Alice to send to Bob, and mode a of state $|\psi_2\rangle$ as the value of $f(x)$ returned by Bob. When $k_1k_0 = 00$, the c and d modes are completely decoupled from the lower circuit. Using our beamsplitter convention, the states are thus

$$|\psi_0\rangle = |0101\rangle \tag{4.1}$$
$$|\psi_1\rangle = B|\psi_0\rangle = \frac{1}{\sqrt{2}}\Big[|0101\rangle + |0110\rangle\Big] \tag{4.2}$$
$$|\psi_4\rangle = |\psi_3\rangle = |\psi_2\rangle = |\psi_1\rangle \tag{4.3}$$
$$|\psi_5\rangle = B^\dagger|\psi_4\rangle = |0101\rangle\,. \tag{4.4}$$

This is the expected result, because c and d form an independent, balanced Mach-Zehnder interferometer, and since the control input to the Fredkin gate is zero, no switching occurs, and the output state is the same as the input. Note that the result is a pure state, and so the photon number measurement result is not stochastic. If the function chosen by Bob is $k_1k_0 = 01$, the result is similar; this time, the phase shift S interacts with the photon input to mode b, giving us

$$|\psi_0\rangle = |0101\rangle \tag{4.5}$$
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[|0101\rangle + |0110\rangle\Big] \tag{4.6}$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big[|1001\rangle + |1010\rangle\Big] \tag{4.7}$$
$$|\psi_3\rangle = S|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big[-|1001\rangle - |1010\rangle\Big] \tag{4.8}$$
$$|\psi_4\rangle = \frac{1}{\sqrt{2}}\Big[-|0101\rangle - |0110\rangle\Big] \tag{4.9}$$
$$|\psi_5\rangle = B^\dagger|\psi_4\rangle = -|0101\rangle\,. \tag{4.10}$$

Both these results are trivial, since whenever $k_1 = 0$, the result returned by Bob, $f(x)$, is *independent* of x.

However, a nontrivial output results when $k_1 = 1$. Consider $k_1k_0 = 10$. Here, Bob's transform $U_{f_{10}} = F$ is a Fredkin gate acting on a, b, and c, and we get

$$|\psi_0\rangle = |0101\rangle \tag{4.11}$$
$$|\psi_1\rangle = B|\psi_0\rangle = \frac{1}{\sqrt{2}}\Big[|0101\rangle + |0110\rangle\Big] \tag{4.12}$$
$$|\psi_2\rangle = U_{f_{10}}|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[|0101\rangle + |1010\rangle\Big] \tag{4.13}$$
$$|\psi_3\rangle = S|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big[|0101\rangle - |1010\rangle\Big] \tag{4.14}$$
$$|\psi_4\rangle = U^\dagger{}_{f_{10}}|\psi_3\rangle = \frac{1}{\sqrt{2}}\Big[|0101\rangle - |0110\rangle\Big] \tag{4.15}$$
$$|\psi_5\rangle = B^\dagger|\psi_4\rangle = -|0110\rangle\,. \tag{4.16}$$

This result can be understood by realizing that if the control signal input to a quantum Fredkin gate is a superposition state, then the outputs will also be superposition states. Thus, the state $|\psi_2\rangle$ returned by Bob leaves y in a superposition state, and since the phase shift S has an effect only when its input is $|1\rangle$ (i.e., not the vacuum), it "filters" out and marks those cases where $f(x)$ has odd parity. This nontrivial result is obtained by virtue of the quantum coherence between all four states. The result for $k_1k_0 = 11$ is similar:

$$|\psi_0\rangle = |0101\rangle \tag{4.17}$$
$$|\psi_1\rangle = B|\psi_0\rangle = \frac{1}{\sqrt{2}}\Big[|0101\rangle + |0110\rangle\Big] \tag{4.18}$$
$$|\psi_2\rangle = U_{f_{11}}|\psi_1\rangle = \frac{1}{\sqrt{2}}\Big[|1001\rangle + |0110\rangle\Big] \tag{4.19}$$
$$|\psi_3\rangle = S|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big[-|1001\rangle + |0110\rangle\Big] \tag{4.20}$$
$$|\psi_4\rangle = U^\dagger{}_{f_{11}}|\psi_3\rangle = \frac{1}{\sqrt{2}}\Big[|0110\rangle - |0101\rangle\Big] \tag{4.21}$$
$$|\psi_5\rangle = B^\dagger|\psi_4\rangle = |0110\rangle\,. \tag{4.22}$$

Note that the output is very different when k_1 is zero or one. Let z be the measurement result for mode d. When $k_1 = 0$, the result is $z = 1$, and Alice's correct conclusion is that Bob's function is type 1. Likewise, when $k_1 = 1$, Alice finds that $z = 0$, and concludes that Bob's function is type 2.

Another way to understand physically what is happening is to reduce the circuit by breaking the abstraction barrier around Bob's apparatus, and taking advantage of the fact that a π phase shift sandwiched between two beamsplitters is just a crossover switch. We consider the $k_1k_0 = 10$ case, where the circuit reduces to become that shown in Figure 5A. We have two interferometers linked by Kerr media; in the bottom interferometer, the photon is split at the first beamsplitter. If it takes the upper path, then it causes a π phase shift in mode c via cross-phase modulation in the first Kerr medium. Alternatively, if the photon takes the bottom path, it also causes a π phase shift in c, this time through the second Kerr medium. Either way, the result is the same; the upper interferometer is unbalanced by π, and thus its inputs are exchanged to give the outputs. This explains why the output is $|\psi_5\rangle = |0110\rangle$ in Eq.(4.16). Note the usefulness of the Everett many-worlds interpretation of quantum mechanics [21] in explaining the operation of this quantum computer. Another interesting observation is that if $k_1 = 1$, then inserting and removing the phase shift S should have the effect of turing k_1 on and off. This effect is the signature of quantum parallelism in our apparatus.

Finally, it is interesting to consider what happens if classical operation of this machine is attempted. If a coherent state $|\alpha\rangle$ is used to represent logical one, and the vacuum $|0\rangle$ as logical zero, the machine will fail in the following way: the measurement results will be independent of whether S is in-place or removed. Consider the $k_1 k_0 = 10$ case, and simplify the circuit to the two circumstances shown in Figure 5. Now, it is well known that the outputs of a beamsplitter fed with a coherent state and a vacuum input are coherent states with half the expected photon number,

$$B|0,\alpha\rangle = |\alpha/\sqrt{2}, -\alpha/\sqrt{2}\rangle\,, \tag{4.23}$$

since this is just the expected classical behavior. In this case, both arms of the lower interferometer will contain the same number of photons, so the photons in mode c will receive the same cross-phase modulation in both cases. When S is in-place, c will get a phase shift once from b and once from a, and when S is removed, c will be phase shifted twice by b. Since the amount of shift is the same in either case, the measurement result is independent of presence of S.

This shows that quantum parallelism does not occur in our machine under classical operation. This is not a surprising result, since a beamsplitter does not create a Schrödinger cat state of $|0\rangle$ and $|\alpha\rangle$ from a coherent state input.

V. ERROR CORRECTION

An important feature of our simple quantum computer is its use of a dual-rail qubit representation. Given correct input preparation, we expect at all times that a single photon exists in *either* mode c or d, but not both; likewise for modes a and b. This feature allows us to detect certain cases when information is lost from the computer, and reject the faulty data. Although this error correction scheme is simple-minded and does not solve the general quantum error correction problem, it is simple to implement, and effective in reducing the probability of error, as we shall see in this section.

Because the machine operates deterministically under perfect conditions, error correction is easy. If the measurement result for the four modes ever changes without any change of the inputs or the switch conditions, then somewhere, a random process must be interacting with the qubits in the machine. For example, measurement of a total of zero or one photons at the output is indicative of a loss process, while measurement of more than two photons suggests some error in preparation of the inputs. Assuming that input preparation is always perfect, we may correct for random errors by rejecting all executions which result in one of $|0000\rangle$, $|0001\rangle$, $|0010\rangle$, $|0100\rangle$, or $|1000\rangle$. We may also reject $|1010\rangle$ and $|1001\rangle$, since we know *a priori* that the scratch-pad (qubit $\{a, b\}$) should remain logically unchanged. When rejection occurs, we perform a re-trial execution.

Let us now consider a specific decoherence model. The Kerr medium used by Bob in his quantum Fredkin gate is experimentally known to be lossy [22], and we may model this by inserting a loss mechanism in modes b and c. Without loss of generality, we consider just the $k_1 k_0 = 10$ case, and imagine having loss occur only during the second instantiation of Bob's apparatus. Specifically, just as before, we have

$$|\psi_3\rangle = S U_{f_{10}} B |0101\rangle \tag{5.1}$$

as the state sent by Alice to Bob in her second communication. We now dismantle Bob's apparatus; in the absence of decoherence, Bob performs the transform $U_{f_{10}} = B_{\rm ab} K_{\rm bc} B^\dagger_{\rm ab}$, where $B_{\rm ab}$ is the usual 50/50 beamsplitter acting on modes a and b, and $K_{\rm bc} = \exp[i\pi b^\dagger b c^\dagger c]$ is the Kerr operator acting on modes b and c. However, we shall consider instead $\tilde{U}_{f_{10}} = B_{\rm ab}\Gamma_{\rm b}\Gamma_{\rm c} K_{\rm bc} B^\dagger_{\rm ab}$, where Γ_i is a *non-unitary* amplitude damping operator acting on mode i. The formal operation of Γ_i is best described by its action on a general single qubit density matrix,

$$\Gamma_i \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \Gamma_i^\dagger = \begin{bmatrix} \rho_{00} + (1-e^{-\gamma})\rho_{11} & e^{-\gamma/2}\rho_{01} \\ e^{-\gamma/2}\rho_{10} & e^{-\gamma}\rho_{11} \end{bmatrix} . \tag{5.2}$$

In other words, Γ_i describes the amplitude damping due to a Caldeira-Leggett type coupling [23] of mode i to the environment, with coupling constant γ. We concern ourselves only with the reduced density matrix of the system here; a good description of this procedure can be found in standard quantum-optics textbooks [24].

The calculation of the output result is straightforward using density matrices. We get

$$|\psi_{3a}\rangle = B_{\rm ab}|\psi_3\rangle \tag{5.3}$$

$$\rho_{3a} = |\psi_{3a}\rangle\langle\psi_{3a}| \tag{5.4}$$

$$\rho_{3b} = \Gamma_{\rm b}\Gamma_{\rm c}\rho_{3a}\Gamma_{\rm c}^\dagger\Gamma_{\rm b}^\dagger \tag{5.5}$$

$$\rho_{3d} = B_{\rm ab} K \rho_{3d} K^\dagger B_{\rm ab}^\dagger \tag{5.6}$$

$$\rho_4 = B^\dagger \rho_{3d} B\,, \tag{5.7}$$

where the density matrix ρ_{3a} describes the input to the loss medium, ρ_{3b} is the input to the Kerr medium (calculated using Eq. 5.2), ρ_{3d} is the output of Bob's apparatus, and ρ_4 is the final output. The diagonal elements of ρ_4 give us the final measurement result probabilities. Physically, we expect errors to occur because the loss of photons results in the possibility of the second Fredkin gate failing to switch. Thus, loss either causes an incorrect total output photon count, or results in the incorrect location of an output photon.

Without error correction, we simply look at the measurement result for mode d. Since the expected result is that $z = 0$ for the $k_1 k_0 = 10$ case, we find that the error probability is

$$P_{\mathrm{NOEC}} = \frac{1}{4}\left[1 + e^{-\gamma} - 2e^{-3\gamma/2}\right] . \qquad (5.8)$$

On the other hand, if we perform error correction by rejecting all illegal results, then the error probability is given by the relative probability of getting $|0101\rangle$ (the wrong answer) to $|0110\rangle$ (the right answer),

$$P_{\mathrm{EC}} = \frac{1}{2}\left[1 - \operatorname{sech}\frac{\gamma}{2}\right] . \qquad (5.9)$$

The dramatic improvement in our error rate given by use of the dual-rail qubit error correction scheme is shown in Figure 6. Work is currently in progress to extend these results to consider other noise sources, such as phase randomization.

It is possible to generalize our results to the N-bit Deutsch problem, using the techniques outlined in the previous two sections, although we shall not do so here. Rather, let us summarize the findings from the study of our simple quantum computer: (1) the concept of quantum parallelism, demonstrated through the simultaneous calculation of $f(x)$ for two values of x, is not in conflict with any fundamental principle of physics, or any fundamental source of noise that is apparent in our system, and (2) rudimentary error correction using a dual-rail qubit representation is simple to apply to a quantum computer, and indeed can be effective in indicating coherence loss or improper input preparation. These advances are hopeful signs of the eventual practicality of quantum computing.

VI. CONCLUSION

The experimental realization of a quantum computer is a difficult proposition. By definition, unitary evolution requires complete isolation from the environment. However, at the same time, it must be possible for qubits to interact with each other, so that information processing can occur. This dilemma goes to the heart of a tradeoff that is central to the practicality of quantum computing.

We chose to use single photons as representations of a qubit, in part because it is easy to create superpositions of single photons using a normal beamsplitter. However, it turns out that it is difficult to find a nonlinear optical material with a $\chi^{(3)}$ coefficient sufficiently strong to allow two single photons to give each other π cross phase modulation. In contrast, it is easy to cross-phase modulate two single electrons, via the Coulomb interaction [25], but difficult to fabricate a 50/50 electron beamsplitter shorter than the dephasing length in a high-mobility semiconductor electron gas. The tradeoff is the interaction strength; it seems that in general, if bits strongly interact, then it is easy to make them process information, but difficult to put them into superposition states.

Another general observation comes from contemplating the structure of our quantum computer. There are three interferometers in this simple one-bit machine! The problem is that quantum computing involves the storage and manipulation of information in canonically conjugate degrees of freedom. For example, in our apparatus, information is encoded both in the photon number (in each mode) and the phase of the photon. Interferometers are used to convert between the two representations. This is fine, in our system, because it is feasible to construct stable optical interferometers. However, if an alternate, massive representation of a qubit were chosen, then it would rapidly become difficult to build stable interferometers, because of the shortness of typical de Broglie wavelengths.

Both of the above problems deal with coherence. There is also the issue of timing. The quantum computer envisioned here is ballistic. Although the machine we present is, in principle, perfectly reversible, we have implicitly assumed that no scattering takes place within the system, because such effects would lead to timing jitter which would cause the malfunctioning of the machine. That is because the logical state of our machine is distributed among four modes, and we cannot deal with effects which cause temporal synchronization to be lost. The only solution we have is that given to us by our simple error correction method; in the event of a detected error, throw out the execution trial and try again.

Despite these problems, we believe that Nature favors quantum computing with single photon states in several ways. First, it is very easy to create superposition states using a beamsplitter. These states have been called Schrödinger kittens because of their robustness compared to macroscopic superposition states which are more massive. Also, transformations such as the phase shift S have simple realizations, because $a^\dagger a$ is the number operator for a single photon, rather than for something macroscopic. These features suggest that single photons (or single electrons) are appropriate physical realizations of quantum bits.

Furthermore, we believe that imminent technological advances in the area of single photonics may provide some impetus to the realization of our machine. In particular, we suggest that the single photon turnstile device

[26] may be the solution for generating a quantum bit source with high spectral purity and a well defined clock. This would give us delocalized states with a high Kerr interaction cross-section, and robustness against timing errors. Also, we hope for a new generation of single-photon detectors, such as the single-photon gate FET [27] and new avalanche photodetectors [28]. Finally, we look forward to new nonlinear optical interactions which may give us single-photon driven switches by coherently converting a photon to and from some other particle (e.g., the exiton-polariton) which has a larger nonlinear interaction strength [29].

Realization of our simple quantum computer using optical components is attractive because of the simplicity of our proposal. Because of mirror symmetry, only one quantum logic gate need be implemented. Furthermore, as a practical initial test of quanutum parallelism (and the feasibility of maintaining quantum coherence through a nonlinear medium), Kerr media with $\chi < \pi$ may be used. In this case, insertion and removal of the phase shift S will still give a statistical signature showing whether classical or quantum operation has been achieved.

Our design of a simple quantum computer has laid a foundation upon which more complicated and general purpose systems may be formulated. By describing quantum computation in terms of the traditional tools of quantum optics, and by introducing a system complete with rudimentary error correction, we have constructed an simple framework for analyzing the impact of decoherence, and evaluating the reality of quantum computation. We hope that our work will lead to a future experiment to demonstrate the practicality of quantum computing.

VII. ACKNOWLEDGEMENTS

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x_1	x_0	f_{000}	f_{001}	f_{010}	f_{011}	f_{100}	f_{101}	f_{110}	f_{111}
0	0	0	1	0	1	0	1	0	1
0	1	0	1	1	0	0	1	1	0
1	0	0	1	0	1	1	0	1	0
1	1	0	1	1	0	1	0	0	1

TABLE I. All possible functions $f_{k_2 k_1 k_0}(x)$ for $0 \leq x < 4$. f_{000} and f_{001} are type 1, while the rest are type 2.

x	f_{00}	f_{01}	f_{10}	f_{11}
0	0	1	0	1
1	0	1	1	0

TABLE II. All possible functions $f_{k_1 k_0}(x)$ for $0 \leq x < 2$. f_{00} and f_{01} are type 1, while the rest are type 2.

FIG. 1. Algorithm for solving Deutsch's problem using a quantum computer.

FIG. 2. Unitary transforms for the components of our quantum computer. The operators a and $a^\dagger$ are the usual annihilation and creation operators.

FIG. 3. Boolean logic (left) and reversible logic (right) circuits for the calculation of the (A) one-bit and (B) two-bit functions $f(x)$. k_0, k_1, and k_2 control the classical switches which determine the function calculated. They are set (secretly) by Bob.

FIG. 4. Complete quantum computer system used to solve the one-bit Deutsch problem. The apparatus in the dashed box is used by Bob to calculate $f_k(x)$, and everything else belongs to Alice. In principle, it is not necessary to send mode d to Bob, although it may simplify the implementation in practice.

FIG. 5. Simplified versions of the quantum computer circuit when $k_1 k_0 = 10$, Bob's apparatus is merged in, and (A) the π phase shift S is in-place, or (B) S is removed.

FIG. 6. Error probability for the final measurement result in the $k_1 k_0 = 10$ case, with and without error correction (lower and upper curves). As loss increases to infinity, the error correction scheme becomes ineffective because the photons become localized in an arm of the interferometer, but for small γ, the improvement is substantial; $P_{\mathrm{NOEC}} \sim \gamma/2$ and $P_{\mathrm{EC}} \sim \gamma^2/16$, where loss is 4.34γ [dB].

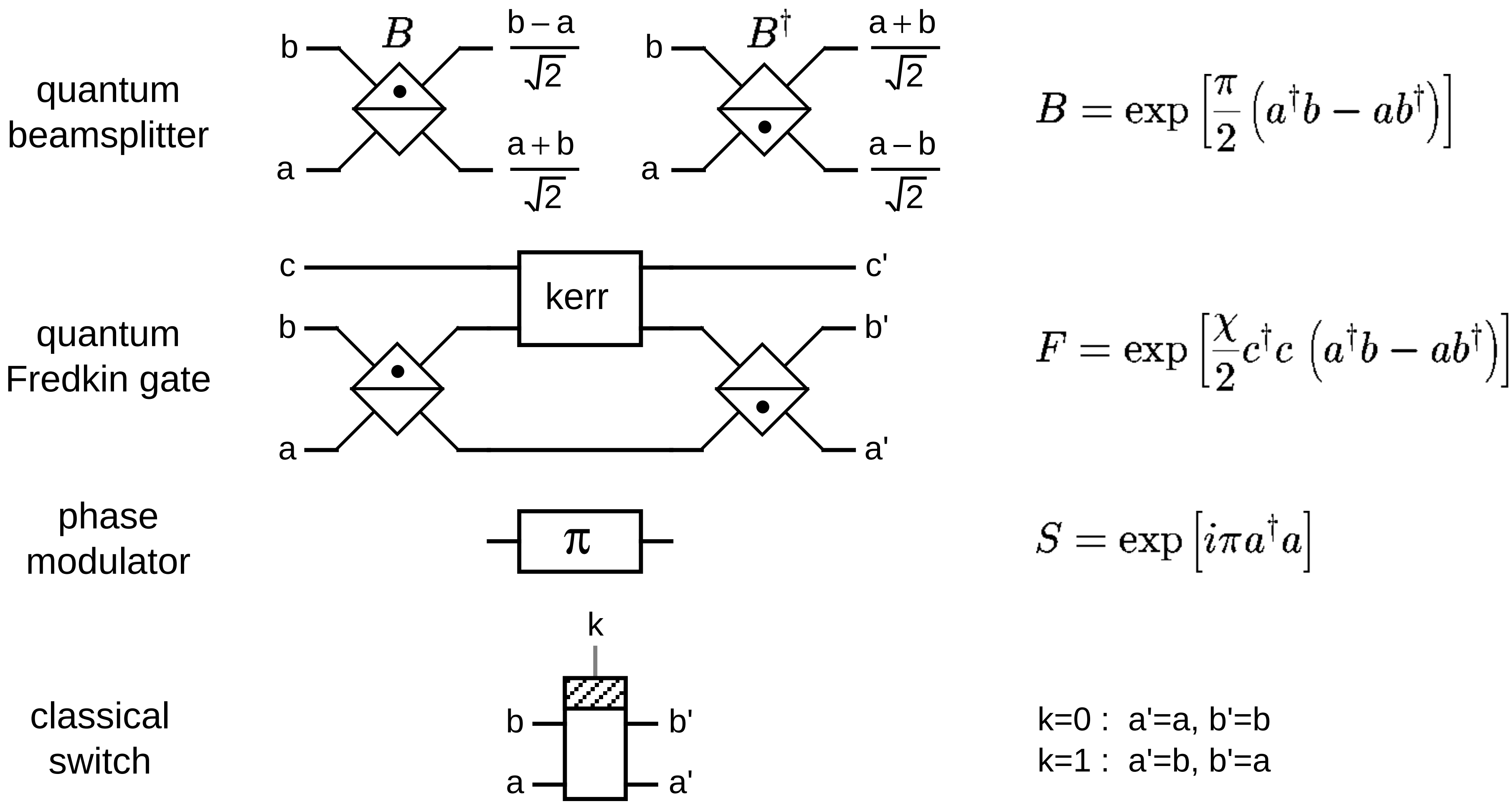

Figure 2

I. Chuang PRA qcomp-28mar95

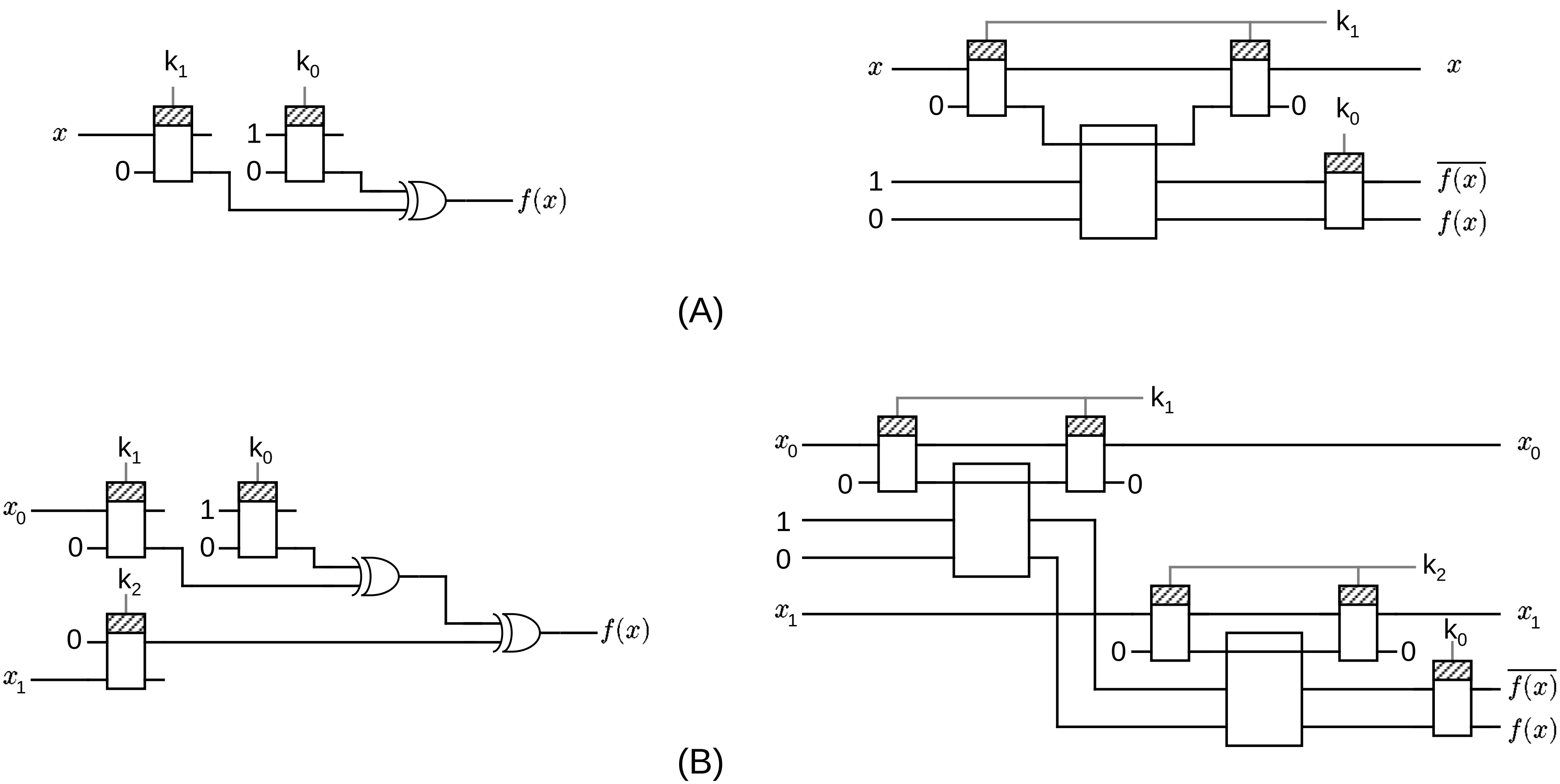

Figure 3

I. Chuang PRA qcomp-28mar95

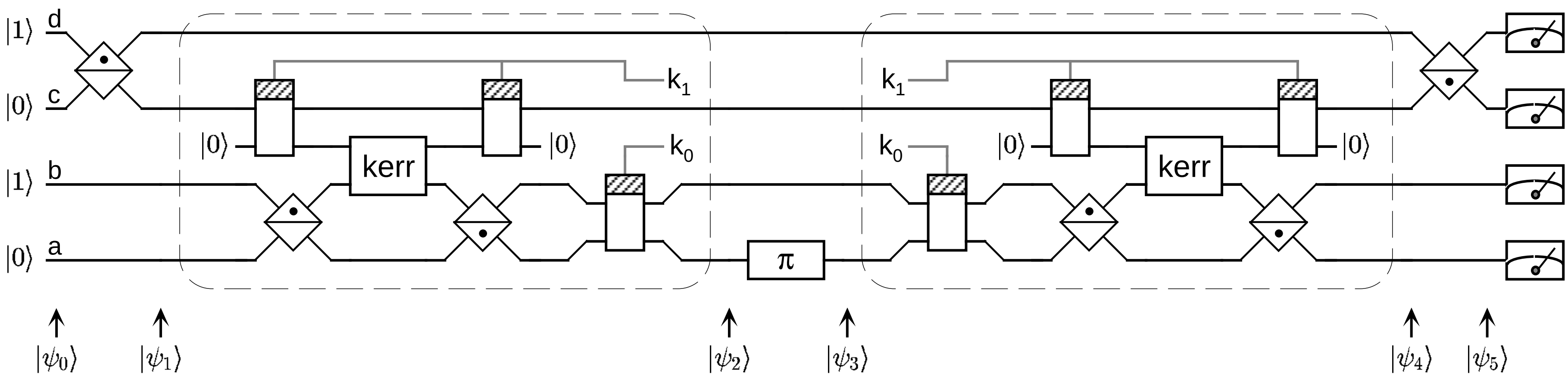

Figure 4

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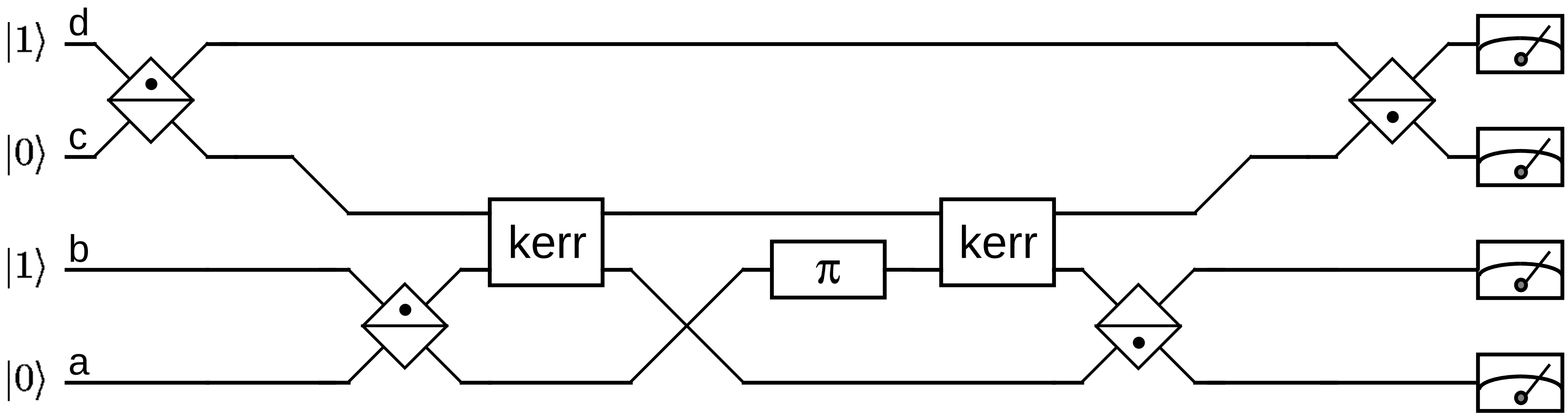
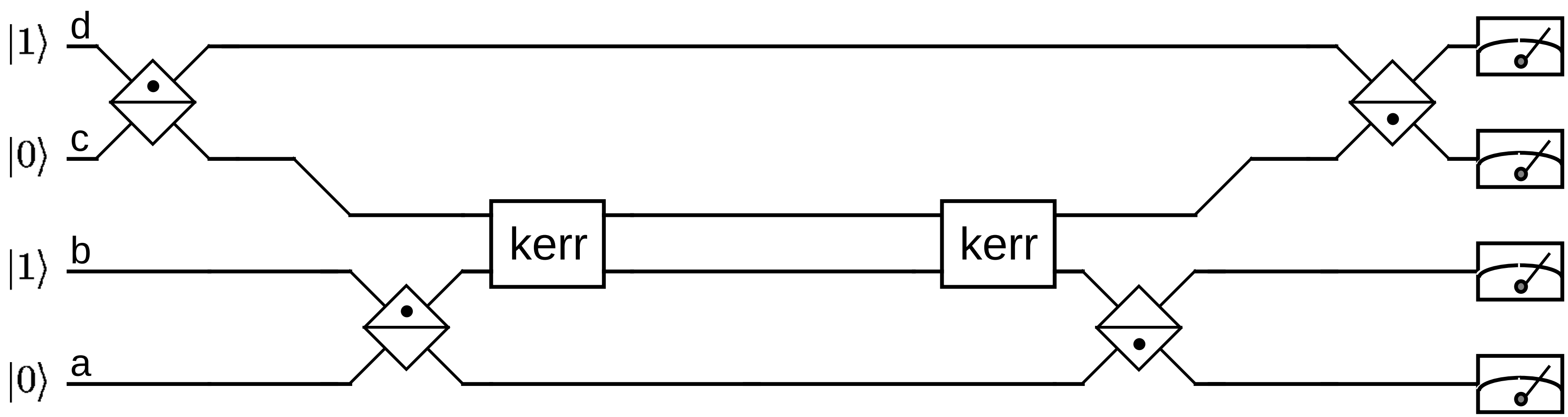


(A)



(B)

Figure 5

I. Chuang PRA qcomp-28mar95

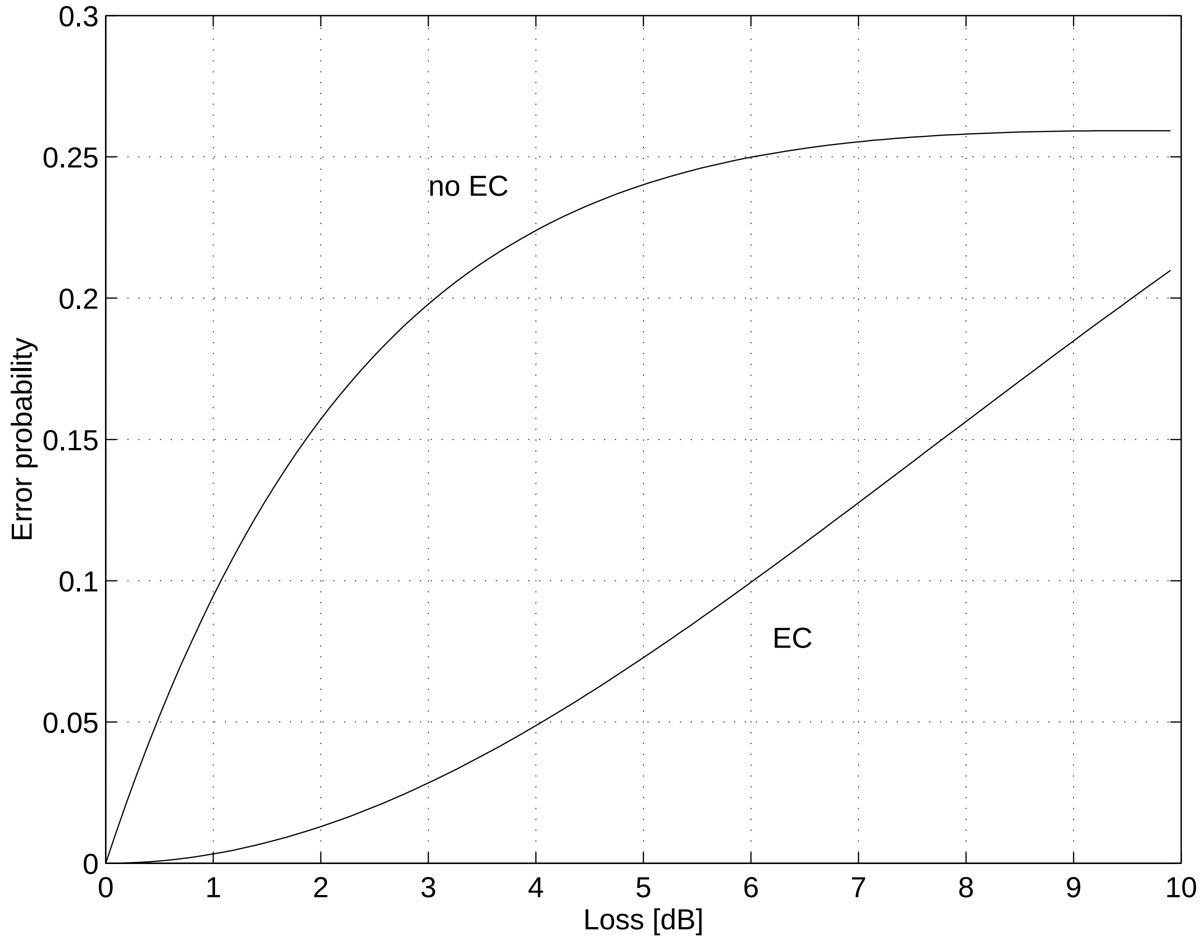


Figure 6

I. Chuang PRA qcomp-28mar95