Quantum random walks

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We introduce the concept of *quantum random walk,* and show that due to quantum interference effects the average path length can be much larger than the maximum allowed path in the corresponding classical random walk. A quantum-optics application is described.

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We introduce in this paper the notion of *quantum random walk,* which is the counterpart of classical random walks for particles which cannot be precisely localized due to quantum uncertainties. A classical one-dimensional random walk is defined in terms of the probabilities for a particle to make a step of a given length to the left or to the right. Quantum random walks are described instead in terms of probability amplitudes. The actual detection process is incorporated into the theory by correlating each possible step to another degree of freedom (say spin), which plays the role of a *quantum coin:* measurement of this observable will select the transition actually undergone. Interesting effects arise when there is a considerable overlap between the probability amplitudes for going left or right. In this case the average displacement of the particle can be well beyond the maximum classically allowed displacement. All these notions are easily generalized to the multidimensional case.

The concept can be exemplified by a spin-$\frac{1}{2}$ particle undergoing a one-dimensional motion, the decision on whether the particle takes a left or right step depending on the outcome of the measurement of the z component of its spin. We emphasize, however, that for our purposes it is not important that the motion occurs in configuration space—thus, if the roles of position and momentum are interchanged, we get a situation typical of a Stern-Gerlach experiment; furthermore, later in this paper we discuss a concrete realization of a quantum random walk in Fock space. Considering for definiteness the configuration-space example, it is easily seen that the time translation corresponding to one step of length l may be represented by the effective unitary operator $U = \exp(-iS_zPl/\hbar)$, where P and S_z are the operators corresponding to the momentum and the z component of the spin, respectively. The eigenstates of S_z are denoted by $|\pm\rangle$, so that $S_z|\pm\rangle = \pm\hbar/2|\pm\rangle$. If the particle is initially in the state $|\psi(x_0)\rangle(c_+|+\rangle + c_-|-\rangle)$, where $\langle x|\psi(x_0)\rangle$ corresponds to a wave packet centered around x_0, and $|c_-|^2 + |c_+|^2 = 1$, then after one step one has

$$|\Psi\rangle = c_-|-\rangle|\psi(x_0 - l)\rangle + c_+|+\rangle|\psi(x_0 + l)\rangle\,, \tag{1}$$

where $|\psi(x_0 \pm l)\rangle$ is centered around $x_0 \pm l$.

Through the time development described above, a strong correlation is established between "right" or "left" and the spin states: for the state described by Eq. (1), a measurement of the z component of the spin (flipping of the "quantum coin") determines, according to the outcome, whether the particle would be described, after the first step, by the state $|\psi(x_0 + l)\rangle$ (if the spin is up) or by the state $|\psi(x_0 - l)\rangle$ (spin down). After measuring the spin, thus determining the new state of the particle, we reestablish the initial condition of the measurement apparatus, and let the state evolve again as described by Eq. (1). It is clear that repetition of this procedure will lead, after N steps, to an average displacement given by $\langle x\rangle = Nl(|c_+|^2 - |c_-|^2)$. These results coincide precisely with those expected from a classical random walk.

A more interesting outcome is obtained by making use of the "multisided" character of quantum coins, and considering a new pair of sides. One measures instead the spin components along a direction (θ, ϕ), where ϕ is the argument of c_-/c_+. The corresponding eigenstates are $|\theta, \phi, +\rangle = \cos(\theta/2)|+\rangle + \exp(i\phi)\sin(\theta/2)|-\rangle$ and $|\theta, \phi, -\rangle = \sin(\theta/2)|+\rangle - \exp(i\phi)\cos(\theta/2)|-\rangle$. Immediately after the measurement, if the spin is found to be $\pm\hbar/2$, the state of the system is, respectively,

$$|\Psi'_\pm\rangle = Z_\pm^{1/2}\left[c_\pm e^{\mp iPl/\hbar} \pm c_\mp e^{\mp i\phi}\tan(\theta/2)e^{\pm iPl/\hbar}\right] \times|\psi(x_0)\rangle\,, \tag{2}$$

where $Z_\pm$ are normalization constants. From these expressions, we see that after the measurement the particle is placed in a superposition of the states centered at $x_0 - l$ and $x_0 + l$. If the spatial width of the state $|\psi(x_0)\rangle$ is much larger than l, we can approximate $\exp(\pm iPl/\hbar)|\psi(x_0)\rangle \approx (1 \pm iPl/\hbar)|\psi(x_0)\rangle$, so that, up to a normalization constant, $|\Psi'_\pm\rangle = [1 + i\delta x_\pm P/\hbar]\,|\psi(x_0)\rangle$, where

$$\delta x_\pm = \frac{c_\pm \mp c_\mp \exp(\mp i\phi)\tan(\theta/2)}{c_\pm \pm c_\mp \exp(\mp i\phi)\tan(\theta/2)}\,l\,. \tag{3}$$

We can see that, so long as $|\delta x_\pm|$ is much smaller than the spatial width of the state $|\psi(x_0)\rangle$, we can write $|\Psi'_\pm\rangle \approx \exp(i\delta x_\pm P/\hbar)|\psi(x_0)\rangle$, and therefore under these conditions $|\Psi'_\pm\rangle$ coincides with the state $|\psi(x_0)\rangle$ displaced by the quantity $\delta x_\pm$. On the other hand, from Eq. (3), it is clear that either $|\delta x_-|$ or $|\delta x_+|$ can be made much larger than l, by a convenient choice of $\tan(\theta/2)$ (of course, they should still be kept much smaller than the initial spread, so that our approximations remain valid). For a sufficiently broad initial wave packet, it is possible therefore to have, say, $|\delta x_-|$ at the same time much smaller than the initial width, and still much larger than l. For instance, we may choose $\tan(\theta/2) = |c_-/c_+|(1+\epsilon)$, with $l(\Delta x)^{-1} \ll |\epsilon| \ll 1$, so that $\delta x_- \approx -2l/\epsilon$, implying

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that $l \ll |\delta x_-| \ll \Delta x$. Note that, in this case, the probability of detecting the system in state $|\theta, \phi, -\rangle$ once it was prepared in the state $|\Psi\rangle$ is given by $P_- \approx |c_- c_+|^2 \epsilon^2$, and therefore we are dealing here with a "rare event." Note also that, for the same set of parameters, we get $\delta x_+ = \left(|c_+|^2 - |c_-|^2\right) l + O(l\epsilon)$, while the probability of detecting the spin $+\hbar/2$ is $P_+ \approx 1 - |c_- c_+|^2 \epsilon^2$. For the average displacement $P_+ \delta x_+ + P_- \delta x_-$ we get of course the same value as before, namely $\left(|c_+|^2 - |c_-|^2\right) l$.

After N steps, if after each measurement we "reinitialize" the measurement apparatus, reestablishing the initial condition of the spin, and if one detects successively the spin component $|\theta, \phi, -\rangle$ (which involves a very small probability), one has for the wave function $\psi_-^N(x) = \langle x|\psi_-^N\rangle$,

$$\psi_-^N(x) \propto \sum_{k=0}^{N} [-(1+\epsilon)]^{N-k} \binom{N}{k} \psi_{x_0}[x - (N-2k)l] , \quad (4)$$

where $\psi_{x_0}(x) = \langle x|\psi(x_0)\rangle$. The cumulative effect after N steps can be numerically evaluated directly from Eq. (4). The results are displayed in Fig. 1, for an initial state given by the Gaussian $\exp(-x^2/2)/\pi^{1/4}$. The important displacement of the distribution after only ten steps, by an amount larger than the original width, and much larger than the maximum classically allowed one, is quite apparent. Of course, there is also a small deformation, which becomes more relevant as the number of steps increases.

These considerations represent in fact an application of the preselection and postselection procedures discussed in Refs. [1–4]: by selectively measuring highly improbable events, it is possible to generate a corresponding highly improbable outcome. The above discussion should make it clear that this result is generated by the interference between the states $|\psi(x_0 - l)\rangle$ and $|\psi(x_0 + l)\rangle$, and is therefore closely related to the quantum properties of the system. In this sense, this effect is quite different conceptually from other possible (and trivial) realizations of "rare events," which can be obtained by starting with special "*ad hoc*" distributions. This would be the case, for instance, if $|\psi(x)|^2$ were a two-peaked distribution, the highest peak being centered around a given point (say x_0) of the lattice and the smaller peak being centered around another point (say x_1), far away from the first one. If the particle is measured around x_1, the wave function collapses into a new state centered around the new coordinate, and therefore with an average position very different from the original one. One should emphasize, in this respect, that our result applies to any sufficiently smooth and wide distribution, and is obtained only when a coherent superposition of two states is produced, as a result of the measurement.

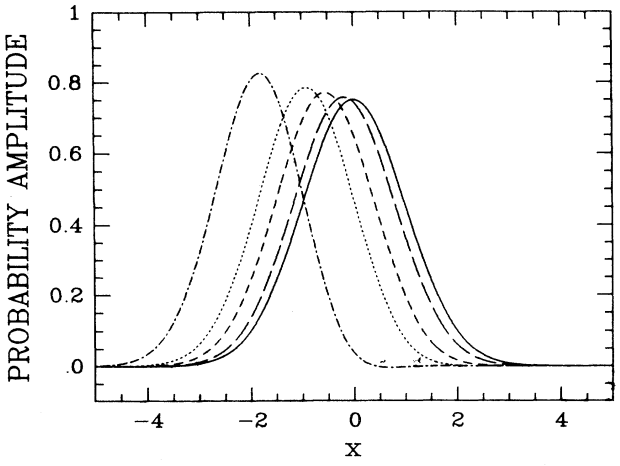

FIG. 1. Probability amplitude distribution for the position of the particle, described initially by the wave packet $\psi(x) = \exp(-x^2/2)/\pi^{1/4}$ (solid line), after one step (long-dashed line), three steps (short-dashed line), five steps (dotted line), and ten steps (dotted-dashed line). Here $l = 0.01$ and $\epsilon = -0.1$. The phases of the successive wave packets have been chosen so that they all have positive maxima.

We show now that a very simple realization of a quantum random walk can be found in the framework of quantum optics. We describe an experiment which results in a drastic reduction (or amplification) of the average number of photons in a cavity, produced by the detection of a single atom, after it interacts resonantly with the cavity field.

Let us consider a cavity in which only one mode is excited, with a photon-number distribution assumed to be such that its average number of photons $\bar{n}$ is much larger than one, and its variance Δn satisfies $1 \ll \Delta n \ll \bar{n}$. At a given instant of time, we inject in the cavity an atom which has two of its levels resonant with the populated cavity mode, and sufficiently far from the other levels so that the atom can be approximately represented by only these two states (two-level atom). We assume for definiteness that the atom is in the upper level (excited state), just before entering the cavity, and that the transit time of the atom in the cavity is much smaller than the lifetimes of both states and the damping time of the cavity (this condition is actually realized in micromaser experiments involving superconducting cavities and Rydberg atoms [5, 6]). Right after leaving the cavity, the atom passes through a region (another cavity) containing an essentially classical electromagnetic field (a microwave field if Rydberg states are involved), which produces a rotation of the atomic Bloch vector. The atom goes then through field-ionization plates [5, 6], which allow one to detect whether the atom is in the excited state.

The initial state of the system composed by the atom and the cavity field is given by $|\Psi(0)\rangle = \sum_{n=0}^{\infty} c(n)|n\rangle|a\rangle$, where $|a\rangle$ denotes the excited state of the atom and $|n\rangle$ is a state of well-defined number of photons (Fock state).

Right after the atom leaves the cavity, the state of the system is [7]

$$|\Psi(\tau)\rangle = \sum_{n=0}^{\infty} c(n) \left[\cos\beta_n |a\rangle|n\rangle + \sin\beta_n |b\rangle|n+1\rangle\right] , \quad (5)$$

where $|b\rangle$ is the lower atomic state, $\beta_n = g\tau\sqrt{n+1}$, g is the atom-field coupling constant, and τ is the transit time of the atom in the cavity. The angle β_n is half the Rabi angle of the atom as it leaves the cavity, when there are n photons in the field. For $n = 0$, we get the vacuum Rabi angle. We note the similarity between (5) and (1),

except for the n dependence of the expansion coefficients in (5).

We let now the atom go through the classical field region, which rotates the atomic state according to $|a\rangle \rightarrow \cos\alpha|a\rangle + \exp(i\phi)\sin\alpha|b\rangle$, $|b\rangle \rightarrow -\exp(-i\phi)\sin\alpha|a\rangle + \cos\alpha|b\rangle$, where ϕ is the phase of the classical field right before the atom enters the interaction region, and α is half the (classical) Rabi angle.

After detecting the atom in the excited state $|a\rangle$, the state of the field inside the cavity is described by

$$|\psi^{(a)}\rangle = \frac{\sum_{n=0}^{\infty}\left[c(n)\cos\beta_n\cos\alpha - c(n-1)e^{-i\phi}\sin\beta_{n-1}\sin\alpha\right]|n\rangle}{\left[\sum_{n=0}^{\infty}|c(n)\cos\beta_n\cos\alpha - c(n-1)e^{-i\phi}\sin\beta_{n-1}\sin\alpha|^2\right]^{1/2}}, \tag{6}$$

where we define $c(-1) \equiv 0$. We show now that, for some special values of the experimental parameters, the average number of photons in this state can be shifted by a quantity as large as the variance of the original distribution, in spite of the fact that only one atom has crossed the cavity.

In order to illustrate this effect in the simplest possible way, we consider the situation in which $\beta_n \ll 1$, corresponding to a weak interaction between the atom and the cavity field, and we assume that the state of the field before the interaction with the atom is a coherent state [8], that is $|v\rangle = \exp(-|v|^2/2)\sum_{n=0}^{\infty}(v^n/\sqrt{n!})|n\rangle$, for which $\bar{n} = |v|^2$. Then Eq. (6) can be written as

$$|\psi^{(a)}\rangle = \frac{[1 - g\tau a^{\dagger}e^{-i\phi}\tan\alpha]|v\rangle}{[\langle v|(1 - g\tau a e^{i\phi}\tan\alpha)(1 - g\tau e^{-i\phi}a^{\dagger}\tan\alpha)|v\rangle]^{1/2}}. \tag{7}$$

The average number of photons in this state can be written as $\langle n\rangle = \bar{n} + 1 + \delta n$, with the first two terms corresponding to the maximum possible classical change, and δn given by

$$\delta n = \frac{\bar{g}^2\tan^2\alpha - 1}{(1 - \bar{g}\tan\alpha)^2 + \bar{g}^2\tan^2\alpha/\bar{n}}, \tag{8}$$

where we have defined $\bar{g} = g\tau\sqrt{\bar{n}}$, and chosen the phase ϕ of the classical field so that $ve^{i\phi}$ is real and positive. Setting $\bar{g}\tan\alpha = 1 - \xi/\sqrt{\bar{n}}$, it is easy to show that δn as a function of ξ has extrema for $\xi = \pm 1 + O(1/\sqrt{\bar{n}})$ (we assume $\bar{n} \gg 1$), where $\delta n = \mp\sqrt{\bar{n}}[1 + O(1/\sqrt{\bar{n}})]$, respectively. Therefore, if $\tan\alpha = (1 \mp 1/\sqrt{\bar{n}})/\bar{g}$, that is, if $\alpha \approx \pi/2 - \bar{g}(1 \pm 1/\sqrt{\bar{n}})$, we can get a displacement of the average photon number of $\mp\sqrt{\bar{n}}$, respectively. These values are way beyond what would be classically expected. Note, in particular, that even though the atom is injected in the excited state, the average number of photons in the cavity can decrease by as much as the variance of the initial distribution, which, for $\bar{n} \gg 1$, represents a displacement much larger than the classically allowed one. We also note that as long as ξ is kept of the order of 1, the variance of the state, which can be exactly calculated, remains practically the same.

In order to get this result, two conditions must be realized: the angle α must be determined with a precision better than $\bar{g}/\sqrt{\bar{n}}$, and the atom must be detected in the excited state, which, for the parameters specified above, involves a very small probability. Under the conditions $\beta_{\bar{n}} \ll 1$, $|\xi| \approx O(1)$, we get that this probability is given by $P_a = (\xi^2\cos^2\alpha + \bar{g}^2\sin^2\alpha)/\bar{n}$. In particular, for $\alpha \approx \pi/2 - \bar{g}(1 \pm 1/\sqrt{\bar{n}})$, we get $P_a \approx 2\bar{g}^2/\bar{n}$, which is much smaller than one.

The exact probability amplitude as a function of n for the state $|\psi^{(a)}\rangle$ can be obtained directly from Eq. (7), and it is displayed in Fig. 2 for an initial coherent state with $\bar{n} = 1000$, for $\xi = 1$ (corresponding to $\delta n \approx -\sqrt{\bar{n}}$), and $\xi = 3$ (corresponding to $\delta n \approx -0.6\sqrt{\bar{n}}$). For these two values of ξ, the exact variations of the average photon number are -31.1 and -17.4, respectively. Note that, for a displacement approximately equal to the variance, the shape of the distribution changes in an appreciable way. The deformation is, however, already small for $\xi = 3$, even though the corresponding displacement is still $\approx 3/5$ of the variance.

Although shown here for a coherent state, the above result can be easily generalized to other field distributions and finite β_n's (which helps to substantially increase the detection probability). In order to simplify the discussion, we set $\phi = 0$ and assume that the amplitudes $c(n)$ in Eq. (5) are real and smooth functions of n, leading to a photon-number distribution such that $1 \ll \Delta n \ll \bar{n}$. Complex amplitudes can be treated in a

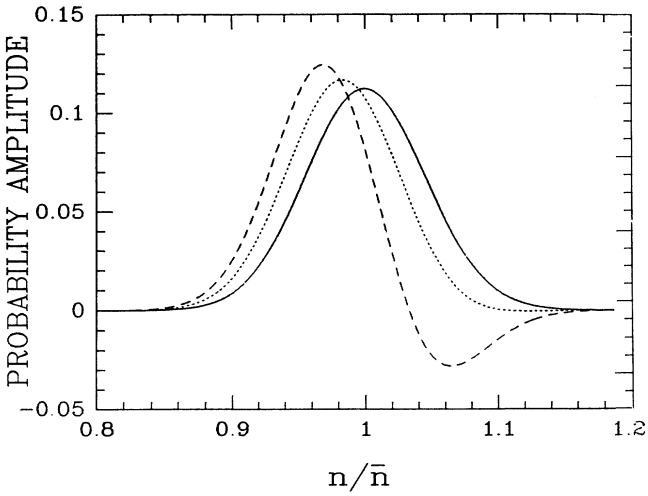

FIG. 2. Probability amplitude distribution for the number of photons in the cavity, when $\bar{n} = 1000$, as a function of the normalized photon number $n/\bar{n}$. The original coherent-state distribution is represented by a solid line. The dotted-line curve corresponds to a reduction of the average photon number equal to 17.4, while the dashed line corresponds to a reduction of 31.1.

similar way (with $\phi \neq 0$), so long as the corresponding phases do not change too fast with n. We also assume that β_n is smaller or of the order of 2π. We can then expand $\cos\beta_n$ and $\sin\beta_n$ in Eq. (6) around $\bar{n}$, and approximate $[(n-\bar{n})/2\sqrt{\bar{n}}]c(n)$ by $-\kappa(\overline{\Delta n^2}/\sqrt{\bar{n}})[dc(n)/dn]$, with κ being a number of the order of one (for a Gaussian probability distribution, we have exactly $\kappa = 1$). We get then (up to a normalization factor) $|\psi^{(a)}\rangle = \sum_{n=0}^{\infty}\{c(n) - \delta n[dc(n)/dn]\}|n\rangle$, where now

$$\delta n = \frac{-[\sin\alpha\sin\beta_{\bar{n}} + g\tau\kappa(\overline{\Delta n^2}/\sqrt{\bar{n}})\sin(\alpha+\beta_{\bar{n}})]}{\cos(\alpha+\beta_{\bar{n}})}. \quad (9)$$

Choosing $\cos(\alpha+\beta_{\bar{n}}) = \epsilon$, with $|\epsilon| \ll 1$, that is $\alpha \approx \frac{\pi}{2} - \beta_{\bar{n}} - \epsilon$, it is easy to see that $\delta n = O(1/\epsilon)$, and therefore we can have $|\delta n| \gg 1$ (with δn either positive or negative), and at the same time much smaller than the width of the distribution, so that $|\psi^{(a)}\rangle \approx \sum_{n=0}^{\infty} c(n-\delta n)|n\rangle$, which represents a displaced distribution. The probability of finding the atom in the excited state in this case is found to be of the order of ϵ^2.

It is possible to imagine an actual experiment, along the lines of the recent micromaser investigations [5, 6], in which this effect could be measured. In order not to have a too small probability and Rabi angle uncertainty, one could deal with a typical micromaser field, with an average number of photons of the order of 100, and have $\bar{g}$ of the order of one. Then the maximum possible displacement of the average photon number would be of the order of ten, which would require a precision in the angle α of better than 0.1 rad. Since the angle α depends on the intensity of the classical field (which can be stabilized to a high degree of precision) and on the transit time of the atom in the corresponding region, this would require a velocity selection of the atomic beam better than 10%, which is easily achievable. Under these conditions, the probability of detecting an atom in the upper state, according to the above approximate calculation, would be of the order of 10^{-2}, well within experimental realization. A numerical calculation for finite Rabi angles, starting directly from Eq. (6), leads to even better results. For $\bar{N} = 100$ and $\bar{g} = 3\pi/2$, we find that if $\bar{N}$ is reduced by 9.4 the atomic detection probability is 0.089. For a reduction of 4.15, the detection probability is as large as 0.4.

The effect analyzed in this paper should be distinguished from other phenomena discussed in the literature, related to the continuous measurement of a quantum system. Thus, in Ref. [9] it is pointed out that the detection of a single atom in the excited (or the unexcited) state, after it leaves the cavity containing the radiation field, may shift the average number of photons in the cavity by a quantity in principle as large as the variance of the original distribution. The change in the distribution and, in particular, in the average photon number results from the filtering of the original distribution due to measurement. Since no quantum coherences are involved in this process, this can be understood in terms of conditional probabilities for detecting photon populations and atomic energies: the new probability distribution is given by the probability of finding the atom in the excited (or unexcited) state, when there are n photons in the cavity, multiplied by the probability of having n photons in the cavity, and summed over all possible values of n. In Ref. [10] the continuous photodetection of a field was analyzed, and it was shown that the average photon number after one count is equal to the previous value minus one plus the Fano factor $\langle[\Delta n(t)]^2\rangle/\langle n(t)\rangle$. Thus, for a Poissonian distribution the average photon number remains unchanged. Both measurement processes are quite different, however, from the one envisaged here. Our proposed scheme relies on the rotation of the atomic Bloch vector after the atom leaves the cavity and before detection occurs. This implies that, after detection, the field is left in a coherent superposition of the original state and the state obtained from it through an overall shift of the number of photons by one. It is the *quantum interference* between these two states that produces a shifted state with a large variation of the average photon number, for *any* smooth distribution. In particular, even a Poissonian distribution may suffer a large shift. Furthermore, our method allows a *precise characterization* of the conditions for getting, after a single measurement, a *shifted* distribution with a large variation of the average photon number.

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[1] Y. Aharonov, D. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).

[2] J. M. Knight and L. Vaidman, Phys. Lett. A **143**, 357 (1990).

[3] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990).

[4] Y. Aharonov, J. Anandan, S. Popescu, and L. Vaidman, Phys. Rev. Lett. **64**, 2965 (1990).

[5] D. Meschede, H. Walther, and G. Mueller, Phys. Rev. Lett. **54**, 551 (1985).

[6] M. Brune, J. M. Raimond, P. Goy, L. Davidovich, and S. Haroche, Phys. Rev. Lett. **59**, 1899 (1987).

[7] M. Sargent III, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addison-Wesley, Reading, MA, 1974), Sec. 14.3.

[8] R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).

[9] P. Meystre, Opt. Lett. **12**, 669 (1987); P. Meystre and E. M. Wright, Phys. Rev. A **37**, 2524 (1988).

[10] M. Ueda, N. Imoto, and T. Ogawa, Phys. Rev. A **41**, 3891 (1990); T. Ogawa, M. Ueda, and N. Imoto, Phys. Rev. Lett. **66**, 1046 (1991).