Neural Graphs & Category of Memory States

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Abstract

The brain as an astonishingly remarkable device has been studied from various angles. It is now well known that neurons are the seat of all activities of the brain function. The dynamical properties pertaining to a single neuron and a collection of neurons may be widely different owing to the clustering properties of a group of neurons. As it can be clearly understood theory of complex physical systems has been more and more employed to study the behaviour of neurons and neuronal circuits. We here mainly discuss neural correlates of memory and cognitive functions utilizing graph theory and ideas from geometry. It has been suggested that stochastic processes being at the helm of affairs in the neuronal level there may exist surfaces to some extent like a hologram for the existence of memory functions.It is also instructive to mention that Amari's developments [1] as regards information geometry has acted as an important inspiration. Unlike some previous analysis categorization of memory from neural perspectives have been reconsidered at the neuronal level. In essence the main point of discussion here has been to give an alternative model of memory where stochastic geometry and algebraic surfaces is an important ingredient.

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1 Introduction

Our current understanding of neuroscience is mainly concerned with the studies of brain as a complex object consisting of a systems of neurons whose proper dynamics is still unknown though several models have been proposed. A brain event may be characterized by an activation which moves through the brain as neurons act sequentially in some physical process.Among these models the Hopfield [2] and its generalizations have been extensively studied and an important result has been obtained like the famous relation $p/N \approx 0.14$ where N is the number of neurons and p are the stored patterns [3]. The general rule of thumb assumed in these models is that the changes in the synaptic strengths are proportional to the correlation between firing of pre and postsynaptic neurons. Since the volume density of synapses in gray matter is brain size independent[4], i.e. $NM/V_g = const$, then we obtain that $M \sim V_g/N$, and as a consequence $p \sim V_g/N^2$. The total number of neurons N is proportional to the total cortical surface area W [5]. The latter scales with the brain volume as: $W \sim V_g^{0.9}$ for large convoluted brains [6]. This leads to the following scaling between the average connectivity p and brain size for convoluted brains.

$$p \sim V_g^{-0.8}. \tag{1}$$

Studies have also indicated [7] that the human brain is a large system, with no more than a hundred specialized modules with different functions. At the fundamental level, the cerebral cortex consists of about 10^{10} neurons that comprise a highly interconnected network. Each cell receives continuously a few thousands of excitatory inputs from other neurons. One of the basic things which is not known is how the cortex, being a mainly excitatory network, prevents the expected explosive propagation of activity and still transmits information across areas. To be more concrete, brain activity happens in bursts, in which pauses, silence or inactivity suddenly and unpredictably are followed by transient activity. The probability Q that the area X connects with the area Y is exactly complementary to the probability that none of the modules in X connects with the area Y, i.e. $(1-\kappa)^{W_0/\xi^2}$, where W_0/ξ^2 is the number of modules in every area. Thus, Q is given by

$$Q \approx 1 - \exp\left(-\frac{aqL_0^2}{\xi^2 K^2}\right). \tag{2}$$

where L_0 is the average length of axons in white matter, K is the number of areas in the cortex, κ is the probability of connection between a given module in one area to another area, ξ is the linear size of a module in the cortex, a is a dimensionless parameter characterizing cortical geometry and a pattern of axonal organization in white matter, q is the probability of sending at least one macroscopic axonal bundle to white matter by a module, W_0 is the surface area of one

cortical area. It is also possible to find an expected number of modules in one area that connect with another area. Assuming that modules are statistically independent, i.e. the probability of sending axonal bundles for a given module does not depend on other modules, the distribution of the number of modules in A reaching B is represented by a binomial distribution. Thus the average number of modules in A connecting with B is given by the product of the probability that a module in A connects with area B and the number of modules in A (W_0/ξ^2), i.e. aqL_0^2/ξ^2K^2. From a dynamical perspective brain dynamics is not different from other natural processes. Nature is clearly non homogeneous and intermittent, the analysis of any natural object reveals an ever surprising amount of details, there is no single relevant scale at which Nature becomes homogeneous. Complexity is this lack of uniformity associated with the scale-free spatiotemporal feature. It is now widely recognized that, under a variety of conditions, non linear systems with many degrees of freedom tend to evolve towards complexity and criticality [8]. It is the interaction of many nonlinear degrees of freedom which produces emergent complex dynamics. Brain activity is eminently spatio-temporal, as such the monitoring of the complicated cortical patterns have greatly benefited from techniques developed in the context of functional magnetic resonance imaging (fMRI). However, the numerical analysis of such spatiotemporal patterns is less developed, lacking mathematical tools and approaches specifically tailored to grasp the complexity of brain cortical activity. One possibility is to get insight from recent work showing that disparate systems can be described as complex networks, that is assemblies of nodes and links with nontrivial topological properties [9, 10]. The brain creates and reshapes continuously complex functional networks of correlated dynamics responding to the traffic between regions, during behavior or even at rest. Some recently studied networks, using functional magnetic resonance imaging in humans [11] has been done. The data is analyzed in the context of the current understanding of complex networks. Some statistical properties of these networks, are, path length and clustering. The path length (L) between two brain sites is the minimum number of links necessary to connect both of them. Clustering (C) is the fraction of connections between the topological neighbors of the sites with respect to the maximum possible. The average clustering of a network is given by $C = 1/N \sum_i C_i$, where N is the number of sites. The scale-free features reflects underlying long range correlations, i.e., brain activity on a given area can be correlated with far away and apparently unrelated regions, something already documented with other technology [12]. One gains a lot of insight into the spontaneous magnetic ordering below a critical temperature if one studies the Ising model, which replaces the rather complex ferromagnetic atom by a simple binary unit interacting with its neighbors. With this analogy it is definitely useful to investigate simple units, which model a few essential mechanisms of neurons and synapses, and to study the cooperative behaviour of such interacting units. It is not obvious at all, whether such a system can store an infinite number of patterns with one set of synapses, learn from examples and generalize. It will

2

be important here to mention that stochastic behavior is crucially important in this analysis and the onset of bifurcation and chaos will be an important determining factor based on the studies of Lyapunov exponents. The neural network of the human brain responds as a unified whole memory bank to a multitude of input signals from the environment and functions with a high degree of robustness and stability. The three aspects of neural networks memory bank are, storage, real-time update and retrieval. The memory is believed to be embedded in the strength of the numerous connections or synapses in the network. Sensory inputs (electrical) produce particular patterns of activity in groups of neurons which then trigger optimal response to the input signal. The cooperative response of millions of neurons to a multitude of input signals has been compared to a very efficient parallel processing computer with neurons and their synaptic connections as fundamental units of information processing, like switches within computers. However, recent studies [13, 14] show that neurons and synapses are extremely complex and resemble entire computers, rather than switches. The interiors of neurons are now known to contain highly ordered parallel networks of filamentous protein polymers collectively termed the cytoskeleton. Information storage, update and appropriate retrieval are controlled at the molecular level by the neuronal cytoskeleton which serves as the internal communication network within neuron. Organization of information at the molecular level in the cytoskeletal network contributes to the overall response of each neuron and the collective activity pattern of neurons then governs the response to the environmental stimuli.

The general awareness or evolution of cognitive functions of the individual may also be governed by the overall background activity pattern of the neurons and their cytoskeletal networks. Coherent signal flow patterns in neural networks may form the basis for general consciousness and response to stimuli (external or internal). Inputs signals trigger spontaneous appropriate coherent pattern formation in the activity of the neurons with implicit spatial correlations in the activity pattern. The time variation of electrical activity of the brain as recorded by the Electro Encephalogram (EEG) exhibits fluctuations on all scales of time, i.e. a broadband spectrum of periodicities (frequencies) contribute to the observed fluctuations [15]. Power spectral analysis which is used to resolve the component frequencies (f) and their intensities, shows that the intensity (power) of the component frequencies follow the inverse power law form $1/f^B$ where B is the exponent. Inverse power law form for power spectra of temporal fluctuations imply long-range temporal correlations. The signatures of short - term fluctuations are carried as internal structures of long - term fluctuations.Neural network activity patterns therefore exhibit long - range spatial and temporal correlations. Such non-local connections in space and time are ubiquitous to time evolution of spatially extended dynamical system in nature and is recently identified as signature of self-organized criticality [16]. Extended dynamical systems in nature have selfsimilar fractal geometry. Selfsimilarity implies that submits of a system resemble the whole in shape. The fractal dimension D is given by $\frac{dlnM}{dlnR}$ where M is the mass contained within

a distance R from a point within the extended object. A constant value for D implies uniform stretching on logarithmic scale for length scale range R. The association of fractal structures with chaotic dynamics has been identified in all dynamical systems in nature. The computed trajectory of time evolution exhibits fractal geometry. The branching interconnecting networks of neurons and intra-neuronal cytoskeleton networks are fractal structures which generate electrical signal pattern with self-similar fluctuations on all scales of time characterised by 1/fB power law behavior for the power spectrum. Such inverse power law form for spectra of temporal fluctuations implies long-range temporal correlations, i. e., long term memory of short term fluctuations or events. Fractal architecture of neural networks supports and coordinates information (fluctuations) flow on all time and space time scales in a state of dynamic equilibrium, now identified as self-organized criticality, is ubiquitous to natural phenomena and is independent of the exact details of the dynamical processes governing the space-time evolution. The physics of self-organized criticality or deterministic chaos is not yet known. Among some other interesting works [17] pioneering work, showed that many functional activities of the brain involve extended assembly of neurons. On this basis, some concepts of Quantum Optics, such as holography, in brain modeling has been developed. Information is indeed observed to be spatially uniform in much the way that the information density is uniform in a hologram [18]. While the activity of the single neuron is experimentally observed in form of discrete and stochastic pulse trains and point processes, the "macroscopic" activity of large assembly of neurons appears to be spatially coherent and highly structured in phase and amplitude . The quantum model of brain proposed [19] is firmly founded on such an experimental evidence. The model is in fact primarily aimed to the description of non-locality of brain functions, especially of memory storing and recalling. The mathematical formalism in which the model is formulated is the one of Quantum Field Theory (QFT) of many body systems. The main ingredient of the model is thus the mechanism of spontaneous breakdown of symmetry by which long range correlations (the Nambu-Goldstone (NG) boson modes) are dynamically generated in many body physics. In the model the "dynamical variables" are identified [20] with those of the electrical dipole vibrational field of the water molecules and of other biomolecules present in the brain structures, and with the ones of the associated NG modes, named the dipole wave quanta. The model, further developed exhibits interesting features related with the role of microtubules in the brain activity [21] and its extension to dissipative dynamics allows a huge memory capacity. The dissipative quantum model of brain has been investigated [22] also in relation with the modeling of neural networks exhibiting long range correlations among the net units. One motivation for such a study is of course the great interest in neural network modeling, in computational neuroscience and in quantum computational strategies based on quantum evolution [23]. Among some other important contributions utilization of functional geometry and an approach of statistical manifold in the brain domain(CNS) [24] has drawn

4

considerable attention in recent years. The proposal is based on the emergence of cognition on the metric and statistical properties of the manifold.

Memory is believed to be a universal feature of the nervous system and exciting results improving our understanding of molecular as well as organizational mechanisms underlying memory have been obtained in recent decades [25]. On the organizational level significant work has been devoted to the study of "brain maps" underlying the ability to recognize patterns or features from a given sensory input [26]. Many intriguing suggestions have been given as to how a memory emerges that is able to extract and recall features from a spatial pattern of neural activity. Time is important in many cognitive tasks but the crucial point is how to represent time, and methods often involve time delays in one form or another. How does a structured memory emerge that can cope with temporal sequences of activity? For example, the information we receive through a temporal sequence of input must at least to some extent be memorized spatially in the neuronal activity pattern. Illustration of these features has been considered in [28] on the fundamental assumptions (a) *Competition* between neural units where excited neural units have an inhibiting effect on other units. (b) *Hebbian Plasticity* is an abstract formulation of long term potentiation depending on pre- and postsynaptic activity: If activity of unit A is followed by activity of unit B the connection from A to B is strengthened [30]. (c) *Recurrent* connectivity opens up the possibility for ongoing information processing in the network by internal feedback. Recently a model for quantum channels with memory has been proposed that can consistently define quantum channels with Markovian correlated noise [31]. The model also extends to describe channels that act on transmitted states in such a way that there is no requirement for interaction with an environment within the model.

So it should be clear that to study the cognitive functions and memory aspects in brain surprisingly geometry may be an major ingredient. As far quantum mechanics is concerned it is not at present clear that at what scale it may be operative [32]. The present work is mainly an approach to the problem of modeling memory and cognitive states from a kinematical or operative point of view, in a sense that we at present being unaware of the exact dynamics of neurons it would be helpful to see and speculate what actually happens. The model we propose here to some extent has a overlap between the holographic and neuromanifold pictures. But the major difference is that they have an algebraic flavour which we will see shortly. The proposal [33] implies that there may be an inherent categorization of the inputs, though it is possibly difficult at this stage to comment on the mechanism. We are of the opinion that there may be different viewpoints and perspectives in studying neurobiology but as far understanding the brain and trying to construct a dynamical theory of the neural mechanism geometry may inevitably play a very important role.

The short paper is organized as follows. In the next section we develop the basic model which we intend to explore. Thereafter we speculate some important connections with the real

systems and try to devise a geometric picture.

2   Neuron Sites and Signal Processing Graphs

Signal processing is inherently stochastic [34]. It would not be an overestimation to state that neural activities do employ signal processing. But the most difficult to answer would be how does it perform?. We show below a diagram to show how impulse can be propagated through the axon and the dendrites.

Loosely speaking, the diagram (B) above resembles a tree graph. In general information proccesing by the neurons can be modeled as a propagation of signal as a transmission line along the nerve cells by a diffusion like process for the voltage transmission

$$b_k \frac{\partial V}{\partial t} = \mathcal{I}_k + \frac{1}{d_k} \frac{\partial}{\partial x}(s \frac{\partial V}{\partial x})$$
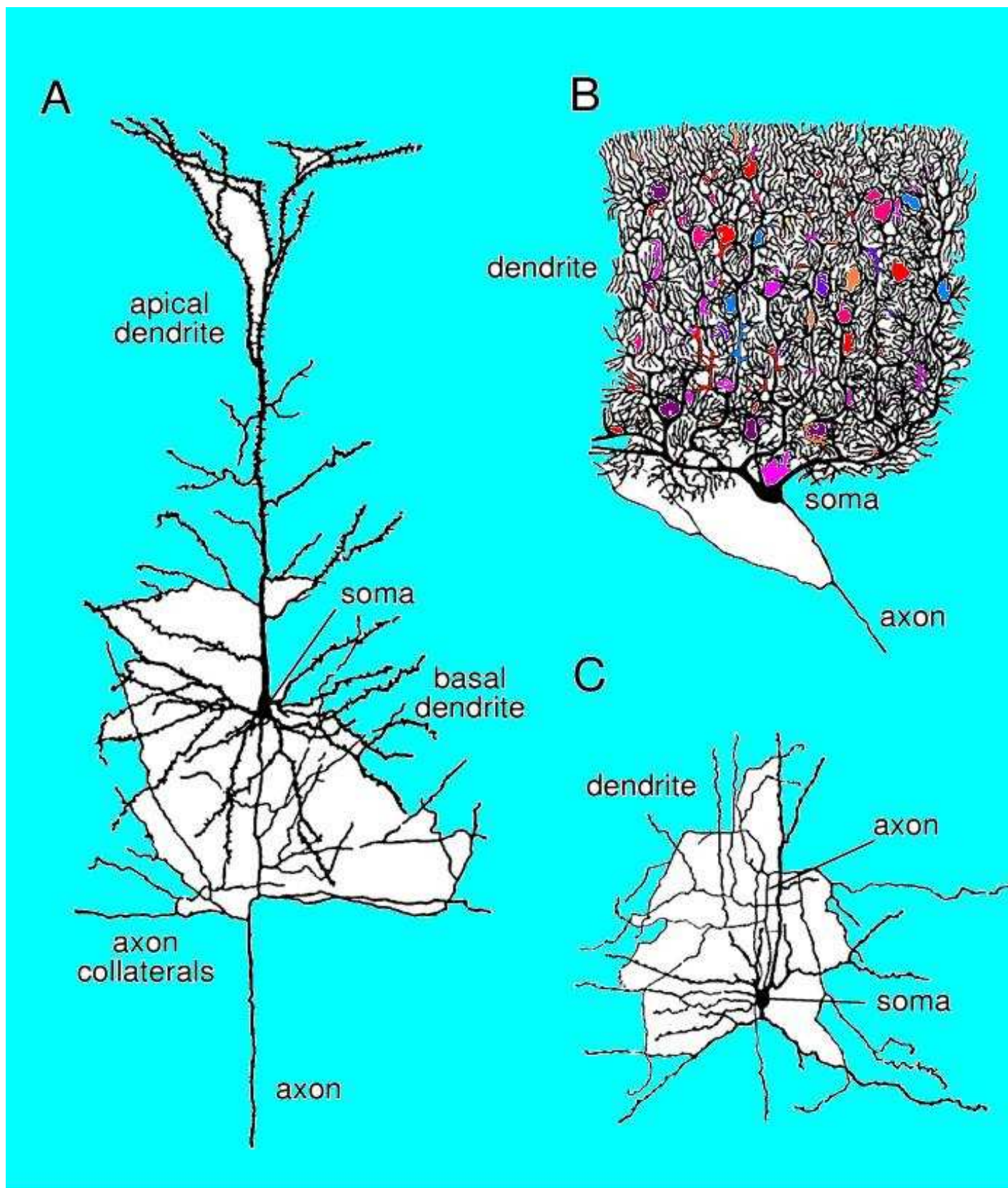
for some choice of the parameters and current. So essentially information processing here arises out of the consideration of the neurons being elements of a circuit. For example the Hodgekin-Huxley model can be reagarded as realising a neuron as an RC circuit on the ionic channels [35]. But the essential point of concern here is whether the processes are stochastic or not. We will delve into these questions in detail soon but at present we would like to mention that there exists some important results which predict neuronal behaviour with some network assumptions with
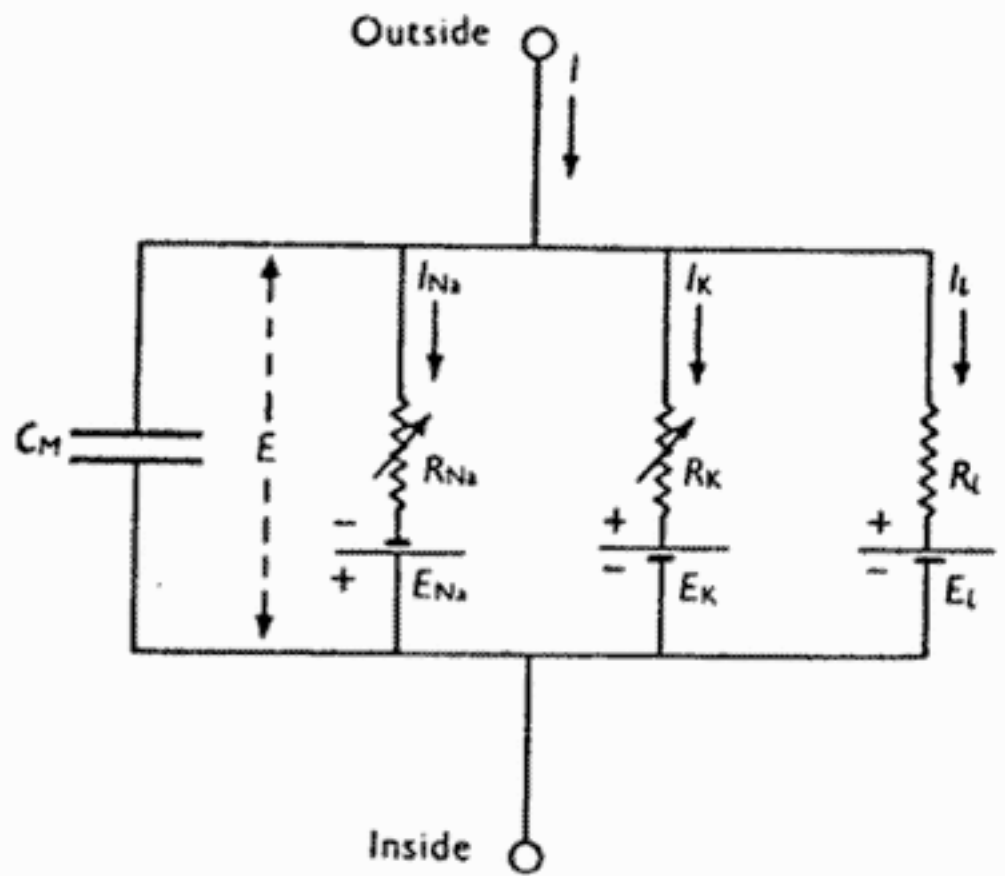
$$V(t) = \int_t^0 \mathcal{K}(t - \zeta)d_k \mathcal{I}_k(\zeta)d\zeta$$

Where $\mathcal{K}(\psi)$ is the kernel and will depend on the chosen model. In general for a spatially structured spike-response model(SRM) neurons the analysis is simpler. This analysis is also very important as our analysis has some structural similarities with it. If we have a large number of SRM neurons arranged on a two-dimensional grid. The synaptic coupling strength A_{ij} is a function of their distance in a functional space of the input impulses λ_i^a. The response of a neuron to the firing of one of its presynaptic neurons is described by a response function ϵ and, the potential is given by a kernel $\mathcal{K}$. The membrane potential of a neuron located at x_i is given by
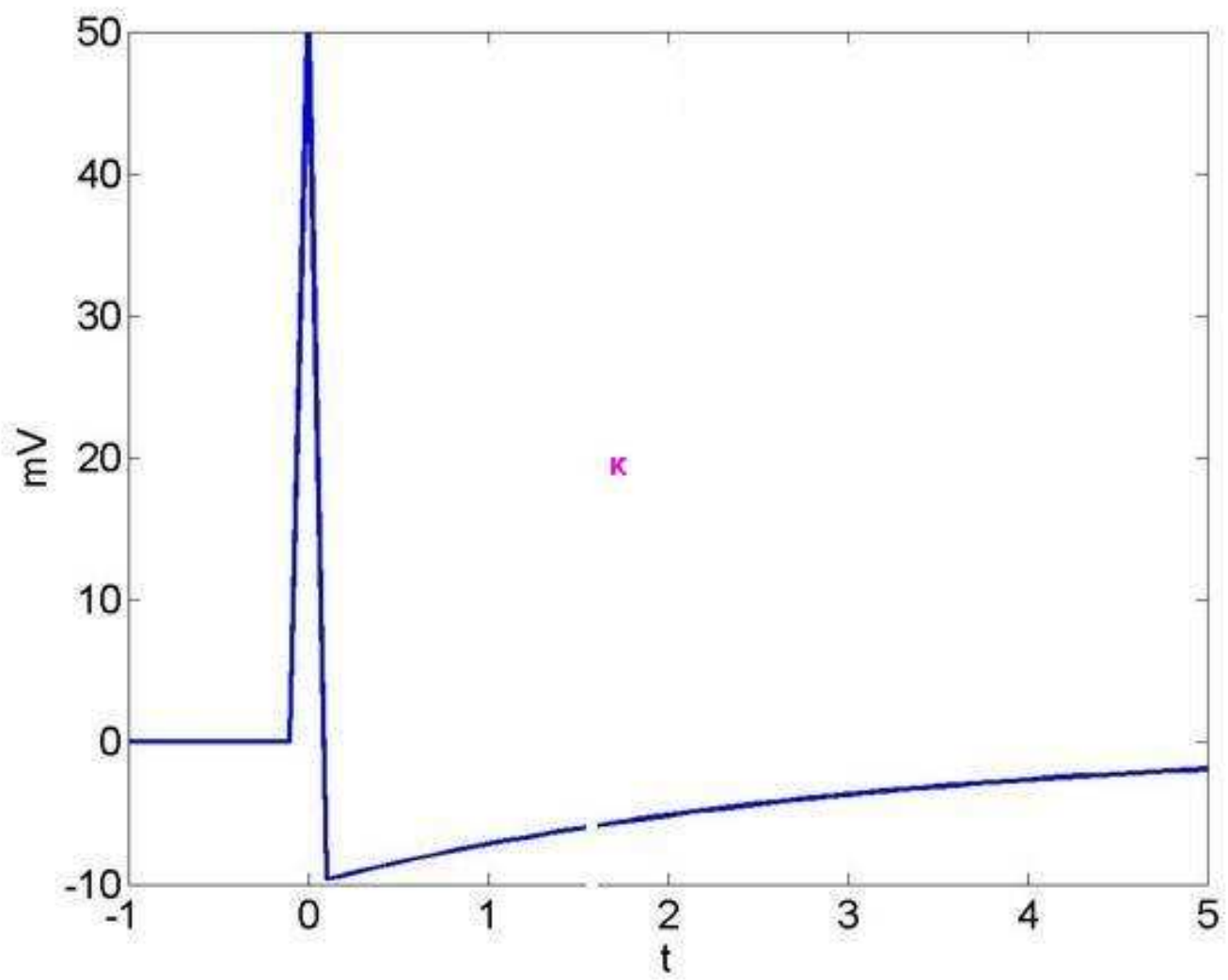
$$V(\lambda_i^a, t) = \int_0^\infty d\psi \mathcal{K}(\psi)S(\lambda_i^a, t - \psi) + \sum_j A_{ij} \int_0^\infty d\psi \epsilon S(\lambda_i^a, t - \psi)$$

with $S(\lambda_i^a, t)$ is the spike train of the neuron. In communication channels every input signal along with a weight factor propagates along with a noise tolerance. There are some ANN models which employ such mechanism. It is important to realize the generation and propagation of a nervous impulse. When the threshold of stimulus is reached the system is switched over

A

apical
dendrite

soma

basal
dendrite

axon
collaterals

axon

B

dendrite

soma

axon

C

dendrite

axon

soma

Circuit depicting neuronal communication via ionic channels

to a new state. So as a whole though it is not clear but it is quite tempting to think that the neuronal processes are essentially monitored by collection of neurons. In our proposal we consider a collection of afferent, intermediate and efferent neurons. It should be mentioned that we at present are ignoring the signal processing aspects of the neurons as we will see shortly that we being mainly interested in memory, though not completely justified those things may be ignored for the present.At this stage the primitive assumptions of the model should be particularly mentioned. It is assumed in our model that the intermediate neurons are large in number in comparison of the other two. It should be mentioned at this point that they are functional attributes and may have some implications on the model after experimental findings. Next we assume that the neurons are arranged and interconnected as if on a lattice, though this may be a debatable issue physiologically, but again we state here that this should not be thought as a physical picture of the descriptions of the neurons but rather a functional point of view. The question that may be asked at this stage whether it is justified to assume a structure may it be functional from the very onset. In this context we wish to state that the mappings of the brain function as has been recently confirmed by some MEG experiments [36] to the cortical area is topological. But we are yet to underpin the nature and form of the mapping.

2.1 Relationships of Input and Output Patterns and Categorization

Let us consider a input pattern [2] λ_i^ε and the output pattern ψ_j^ε The meaningful question which are relevant to be asked for this is the probability $P(\psi_j, \lambda_i)$ with its usual connotations. As is well known from statistical mechanics the average information content for this given configuration is evaluated to be

$$\langle I(\psi, \lambda) \rangle = \lim_{n \to 0} \frac{1}{n} \langle \int d\psi d\lambda P(\psi, \lambda)([\frac{P(\psi, \lambda)}{P(\psi)}]^n - [P(\lambda)]^n) \rangle \tag{3}$$

The above integrals for large neuronal systems is obtained by saddle point evaluation and they may imply a correlation with the signals.

Now as we have previously assumed about or proposed model regarding neurons as input output systems on lattice sites, the essential ingredient of our assumption is that cognitive aspects are a result of dynamics and clustering of neurons in various alignments and arrangements. To depict the picture of the model in detail the lattice sites are occupied by synapse with weights A_{ij} determined by the position of the lattice sites. We think of the neurons as functionally connected together by the synapses, situated at different sites. Now any sensory experience as an input (λ_k^a), from a dynamical point of view gives rise to a spike (e_{ijkl}) (connecting the ij'th site to the kl) which interconnects a group of neurons from the afferent to

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[2]though we neglect here some crucial questions to be asked here as how can we say an initial input to be a pattern before the information is processed

the intermediate and again with a variety of possibilities connects the efferent ones which gives rise to outputs (ψ_k^a). In general the probability distribution of a graph is associated with an input and output state. So in general for a path from the effectors to the affectors via the intermediate layer of neurons will in general be composed of combination of these paths. So rigorously speaking the probability density $P(\psi_j, \lambda_i)$ associated should be taken between all the n (say) input states. If we label the sites with indices then the actual probability density for a effector to affector takes the form of

$$
\mathcal{P} \;=\; \begin{pmatrix} p_{11} & p_{12} & \cdots & \cdots & p_{1n} \\ p_{21} & \cdots & \cdots & \cdots & \cdots \\ p_{31} & \cdots & \cdots & \cdots & \cdots \\ p_{m1} & \cdots & \cdots & \cdots & p_{mn} \end{pmatrix}
\tag{4}
$$

Here we signify p_{ij} as the probability density associated with a chain of the graph, connecting the ij site. In our arguments below though we will mainly be concerned with the probability density function $\mathcal{P}$ disregarding the chain probability densities. It should be clarified at this stage that as wee will see in our development that the neuronal processes do behave as a Markov chain, and the states are given by an activation map $\varphi : \lambda_k^a \to x$. But effectively here we in this paper have made a simplifying choice on the function and made it an identity. [3]So in this model essentially any input as wee see can be thought of as a possible groups of neurons attached with proper weights of synapses at the sites, and the outputs do form sensory experiences mainly memory from our perspectives. Here it should be mentioned that we depict the model here mainly which gives rise to cognitive states such as memory, and ignore some other important aspects such as motor outputs corresponding to the inputs. So importantly here we assume that any experience is basically a network of neurons on the functional lattice space and they can be thought as graphs.We give in Fig (1) a simple diagram of a model graph of neurons for a very low number of sites. It should be noted that there are always possible to have different graph for the same input and outputs, thereby giving rise to the notion of probability distribution for the a specific graphs for a given input-output. In the figure however we have shown some graphs with different input and outputs. The essential point here is that we consider the neural circuits as digraphs $\mathcal{G}_{i,j}$ with n vertices and m edges. We will later try to construct adjacency matrices corresponding to these graphs to see how they can give nontrivial conclusions regarding a network. We assume here that the graphs may not be unique in a sense that corresponding to a group of similar set of experiences we may charecterize the same graph. This can be labelled as one of the limitations of the model, but it is emphasized here that by introducing more parameters in the function space this problem may be resolved. For example we may assume that the neurons have some extra attributes attached to them. So the

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[3]Various models of brain wave may be analogous to this idea, it may be also thought the waves to behave as solitons [37].
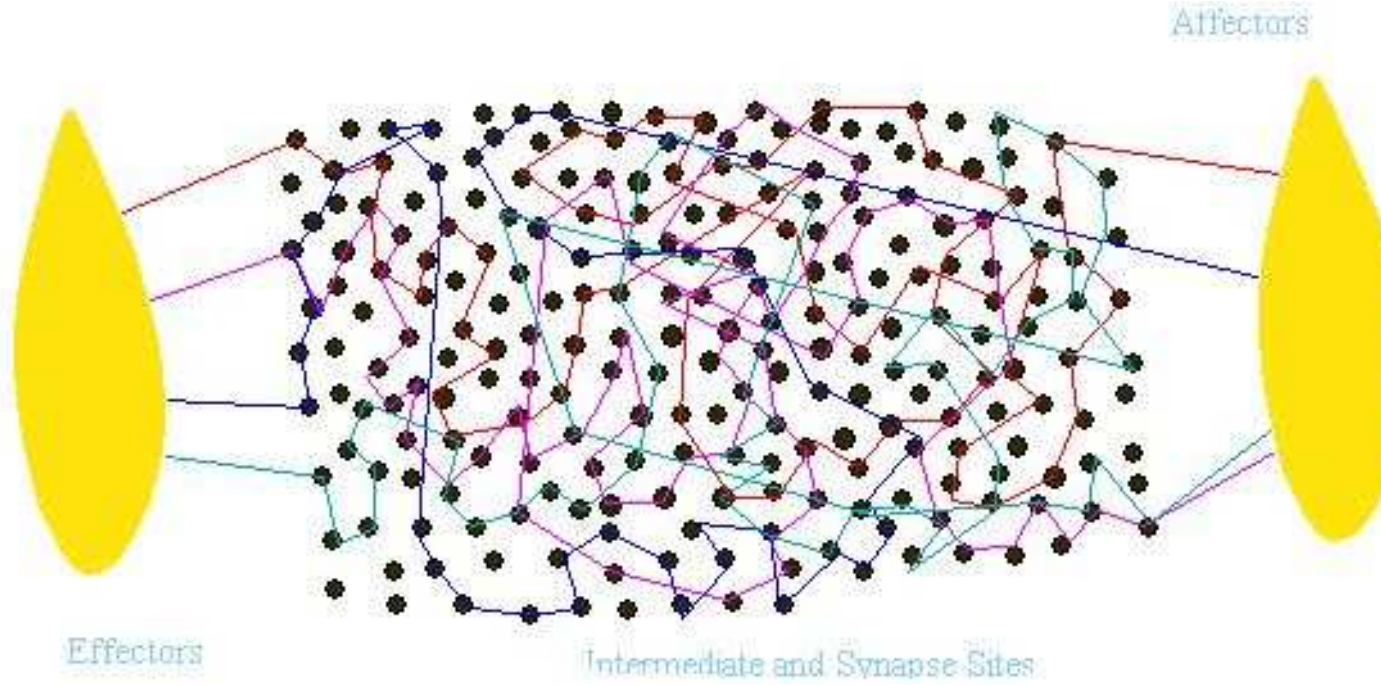
Figure 1: Neuronal Graphs

functional neuron space is mainly composed of graphs which give rise to memory and related cognitive phenomena corresponding to a set of experiences. The graphs can also be thought of arising from thinking the cognitive and memory states to be objects $\mathcal{M}$ which consists of (λ_k^a, ψ_k^a) and the two maps which consists of A_{ij} and e_{ijkl}. So this can be represented as

$$\lambda_k^a \overset{A_{ij}, e_{ijkl}}{\Longleftrightarrow} \psi_k^a \tag{5}$$

Here we have imposed a bijective relationship among the states to signify the feedback mechanism, though the exact relationship of this is not clear in this case. Now if the memory states $\mathcal{M}$ be envisaged as a category with the above objects and maps in it (precisely speaking we have to define the morphisms whose full understanding will require some understanding of the nature of neuronal dynamics) then (5) can be thought as a graph, which are irreflexive directed multigraphs. Thinking the input and outputs to be a space disregarding the internal structure, for each element λ_k^a in the input space we may draw an arrow from this point we get $F(A_{ij}, e_{ijkl})\lambda_k^a$ (where F is an arbitrary map in the category) which will be essentially a point in the output space. The important point here is that the map should preserve the structure of the graph. There are some pertinent points to be mentioned here, first of all in a dynamical point of view, we have the input space of all possible different possible states of the system and the endomap δ of the space which takes each state λ_k^a in which the system will be after a time evolution. For example if we think of an object in $\mathcal{M}$ as a machine the input space is the set of all possible states in which the machine can be and δ gives for each state the state in which the machine will be if the button is pushed once. So from these arguments the idea of

neuronal graphs seems plausible. But as we will see shortly the graphs will have to obey the constraints of neurodynamical equations and this may give rise to interesting conclusions as regards the interpretation and the geometry of the functional space. The functional connectivity of a neural system can be computed by from the covariance matrix of the graph concerned [38]. As it can be realised the categorisation of input patterns do have a reflection in the graphs which we have constructed. There are some interesting experimental results observed [39, 40] on macaque monkey and cat which shows that connectivity patterns have very striking overlaps with graph theoretical analysis of cortical areas.

2.2   Neurodynamics

It is already established that neurobiological processes may be described by the reaction and diffusion models [41] In this context it will be worthy to mention that the Fisher-Burger's and Fokker-Plank equations are well known partial differential equations which govern a wide variety of physical systems. The basic model we want to describe here is based on these prescriptions. We assume here that given an input there is a probability that some neurons will communicate to form a graph to yield a particular output. As it is quite well known that the synaptic weights A_{ij} do change, by the process of learning whereby we get the evolution

$$\hat{A}_{ij} = A_{ij} + \sum e^{ijkl} l_{ij} f(\lambda_k^a, \psi_k^a)$$

where l_{ij} can be labelled as learning parameters and β^k to be the strength by which the inputs are associated with the synaptic weights by the neurons for some function. We also assume that the inputs and outputs should be related with each other after the synaptic weights and the neuronal strengths are integrated out. In other words as wee will see below the models are stochastic there are probability distributions associated with a neuronal graph for a particular input with the corresponding output. The main task for us is to find out the probability distribution, which is most difficult to find out in more general circumstances. We will try to find its nature for some simple situations. So given this we postulate the following relation between the output and the input states.

$$R_n(\psi_k^a) \;\; = \;\; \mathcal{P}(\lambda_k^a) + \sum q_n(g(\lambda_k^a)) \tag{6}$$

where $\mathcal{P}(\lambda, t)$ is the Probability density function of the graph associated with a corresponding input output and the synaptic wights and neuronal spikes, R_n, q_n are assumed to be polynomials of some general function of the output and input states respectively in the simplest approximation. We would like to state here that in analysing the dynamics of the graphs an appropriate spiking function of the neurons has to be incorporated, for example for a stimulus intensity $g(\lambda)$ the spiking function could be chosen as to be the Naka-Rushton function

$$e_{ijkl} \;\; = \;\; \frac{s_{max} g(\lambda)}{g_{1/2}(\lambda) + g(\lambda)}; \quad g \geq 0 \tag{7}$$

$$= 0 \quad otherwise$$

Here $g_{1/2}(\lambda)$ is the stimulus intensity which produces half the maximum firing rate s_{max}. Now the Fisher-Burger [42] scheme gives us a tool to find out a specific differential equation for the probability distribution function. It is quite obvious that the probability distributions will generally be dependent on the inputs and the synaptic weights apart from being time dependent, and we here in this scheme assume that the synaptic weights A_{ij} do depend on the inputs and the probability distribution,the inputs in turn by a feedback mechanism may also depend on the weights and some noise parameters all with appropriate couplings. So the basic equations are given as follows.

$$
\begin{aligned}
-\frac{\partial A_{ij}}{\partial \lambda_k^a} &= \mathcal{P} + \beta f(\lambda_k^a) \\
\frac{\partial A_{ij}}{\partial t} + \mathcal{P} A_{ij} &= -d \frac{\partial \mathcal{P}}{\partial \lambda} \\
\frac{\partial \lambda_k^a}{\partial t} &= -\frac{1}{2} \gamma e_{ijkl} f(\lambda) + \theta \nu_i + \phi A_{ij} + \zeta \psi_k^a
\end{aligned}
\tag{8}
$$

The last equation has a discrete analogue for gradient descent algorithm in ANN's, which is given by

$$\lambda_i^a(t+\epsilon) = \lambda_i^a(t) - \frac{\epsilon}{2} \nabla f_i + J g(\lambda_i^a)\nu_i(t)$$

Here it should be noted that neuronal spikes are assumed to be nondynamical, though this may seem to be an idealisation, it will be seen that the known models to comply with this. Now after some straightforward algebraic manipulations (8) reduces to

$$\frac{\partial \mathcal{P}}{\partial t} - \frac{\partial}{\partial \lambda}(\mathcal{P} A_{ij}) - d \frac{\partial^2 \mathcal{P}}{\partial \lambda^2} = -\beta f^{'}(\lambda)\{-\frac{1}{2}\gamma e_i f(\lambda) + \vartheta \nu_i + \phi A_{ij} + \zeta \psi_k^a\} \tag{9}$$

It should be noted here that (9) reminds us of the basic aspects of population modelling of statistical neurodynamical equations [43]. So essentially by some assumptions on the neuronal variables and forming a set of coupled differential equations between them, we are able to produce the differential equation for the probability density function for the graphs.The equation can be identified with the Fokker-Plank equation in the lower order which is given by the

$$\frac{\partial}{\partial t}\mathcal{U} = -\eta \frac{\partial}{\partial \lambda} a_1 \mathcal{U} + \frac{\eta^2}{2} \frac{\partial^2}{\partial \lambda^2} a_2 \mathcal{U} \tag{10}$$

Here a_1, a_2 drift and diffusion coefficients and $\mathcal{U}$ is the probability distribution.With an appropriate choice of the the coefficients and functions f the equations (9) and (10) can be shown to be equivalent. Though we should like to emphasize that $\mathcal{P}$ in our case denotes the probability density associated with a graph determined by the synaptic weights and neuronal spikes. In this context it should be mentioned from the form of (9) that as we has envisioned about the independence of weights and spikes in the input output evolution, this may not be achieved in

this form without some furthur assumptions on those variables. Now it will be realized that
(9) with proper choice of variables and parameters with some simplifying assumptions, take the
form of diffusion equation given by

$$\kappa \frac{\partial^2 \mathcal{P}}{\partial \lambda^2} = \frac{\partial \mathcal{P}}{\partial t}$$

Now in finding the elementary solutions of the diffusion equation we get

$$\mathcal{P} = \frac{1}{4\pi\kappa t} exp\{-(\lambda - \xi)^2/4\kappa t\}$$

Now given the form of (9) it is clear that the solution will be nontrivial and this stage to
make the probability density function to be dependent on the inputs only we require to impose
constraints on the weights A_{ij} and the spikes e_{ijkl}. So in our case the probability density
function of the graphs the solution will take the form

$$\mathcal{P} = \mathcal{C}\mathcal{F}(\lambda)exp\{-(\lambda - \xi)^2 a/t\} \tag{11}$$

It should be realized that the probability density function satisfies the information geometric
constraints as has been postulated in [44]. To give a geometric flavour in the interpretation
of the probability density function, we state the variant of maximum-minimum principle and
postulate that it may give rise to the information flow in the mental state space.

Theorem:

If we consider $\mathcal{P}(\lambda, t)$ to be a continuous function of its arguments and is a solution of the
diffussion equation, then $\mathcal{P}(\lambda, t)$ attains it's extremum at the boundary.

Now we would like to analyse the evolution on graphs for the inputs by an alternative pro-
posal [45] by means of the Frobenius-Perron(FP) operator and try to construct a corresponding
reachability matrix. At the first place corresponding to each directed edge joining two lattice
points we associate a vector $\mathcal{T}$. We define the FP operator $\hat{\mathcal{O}}$ on the vector by the following
rule

$$\hat{\mathcal{O}}\mathcal{T}(e_{ijkl}, A_{ij}) = \sum_p \mathcal{P}\mathcal{T}(\phi^{-g(\lambda)}(e_{ijkl}, A_{ij}))$$

where the sum is computed over all the possible paths. Taking the Laplace's transform on the
FP operator by assuming a Poisson distribution of the paths on the functional space over the
edges of length l and the learning parameters l_{ij}, the analysis of [46] gives rise to

$$\sum \int_0^\infty e^{-s\lambda} \frac{(s\lambda)^n}{n!} \hat{\mathcal{O}}\mathcal{T}(\phi^{-g(\lambda)}(e_{ijkl}, A_{ij}))d\lambda$$
$$= \int_0^l e^{-s\lambda}(s\lambda)^n \mathcal{T}(e_{ijkl} + l_{ij}log\lambda, A_{ij}))d\lambda \tag{12}$$
$$+ \sum_{n=1}^\infty \sum_p \int_{l-a\sum e_{ijkl}}^{l+a\sum e_{ijkl}} e^{-s\lambda}(s\lambda)^n \mathcal{T}(e_{ijkl} + l_{ij}log\lambda, A_{ij} + l^n))d\lambda$$

15

The second integral in the rhs of (12) can be written in a tractable form by change of variables and identifying

$$\mathcal{Q}_{ee'} = U(\lambda)\mathcal{P}(s\lambda)^n e^{-sl}$$

for some choice of the function U. Thereby it will be seen that the sum over all the paths can be implemented by the matrix $\mathcal{Q}^n$. It can be argued that $\mathcal{Q}$ may be a measure of the connection between the edges via the sites [47] from which the covariance of the neural system can be calculated which will give us a clue about the statistical information variables of the system. So essentially the matrix $\mathcal{Q}$ encodes a great deal of information about the neuronal systems which crucially depends on the probability distribution of the graph.Let us see how much we can proceed in finding out some solutions to the equation (9). Let us try to construct the states in thinking that states are formed in a stationary equilibrium in a sense that we see the evolution of the states after long time has passed after the input is applied. So these solutions in this case may denote the permanent memory states of a neurological system. In finding out the stationary states of the equation we make a proposal that under some suitable choice of the variables, functions and initial conditions (9) can be recast in the form

$$\frac{\partial}{\partial\lambda}\left[\mathcal{A}(\lambda,\mathcal{P}(\lambda))\frac{\partial\mathcal{P}(\lambda)}{\partial\lambda}\right] = G(\lambda,\mathcal{P}(\lambda)) \tag{13}$$

We do assume some strict restrictions on $\mathcal{A}, G(\lambda, \mathcal{P}(\lambda))$ for the model to bring some interesting conclusions. Solvability of the above equation (13) is equivalent to the solvability of the generalized Hammerstein equation [48]

$$\mathcal{P}(\beta) + \int \mathcal{K}(\lambda,\beta;\mathcal{P})G(\lambda,\mathcal{P}(\lambda))d\lambda = 0 \tag{14}$$

where $\mathcal{K}(\lambda, \beta; \mathcal{P})$ is the kernel of the evolution. But the essential point here after this identification is that the solutions of this differential equation are known to have solutions in $L^2[0,1]$ which may be identified with a reflexive Banach space with a finite Borel measure on it [49]. In that case we may like to loosely at this stage identify the neuronal states to be elements of a statistical functional space which as we saw are related by the the probability density function $\mathcal{P}$. Before discussing the geometry of the state space, in this context it is noteworthy to mention the saccade model in context of vector averaging can be applied in this case or not. This case has been analyzed in reference to investigations of visual stimuli and formation of images off geniculate [50]. We propose that the state space geometry can be modeled from of a neural field, $\hat{\mathcal{J}}$, whose evolution for a specific probability distribution should be expected to give rise to the consequences. We state below the basic equation which may give rise to the evolution of the neural field.

$$\dot{\hat{\mathcal{J}}}(\lambda,t) = \mathcal{P}(\lambda,t) * X[\hat{\mathcal{J}}] + \eta(\lambda,t) \tag{15}$$

Here $*$ denotes the convolution, and X is a functional of the neural field operator.

3   Geometry of the functional space and memory states

Before discussing the implications of the previous analysis and the geometry of the graph space let us turn to a simple problem in robotics.

In the figure below there are 3 linked rods of lengths of 6, 4, 2 respectively with specific coordiantes. The states(positions) of the arm are determined by the solution in R^6 to the following polynomial equations.
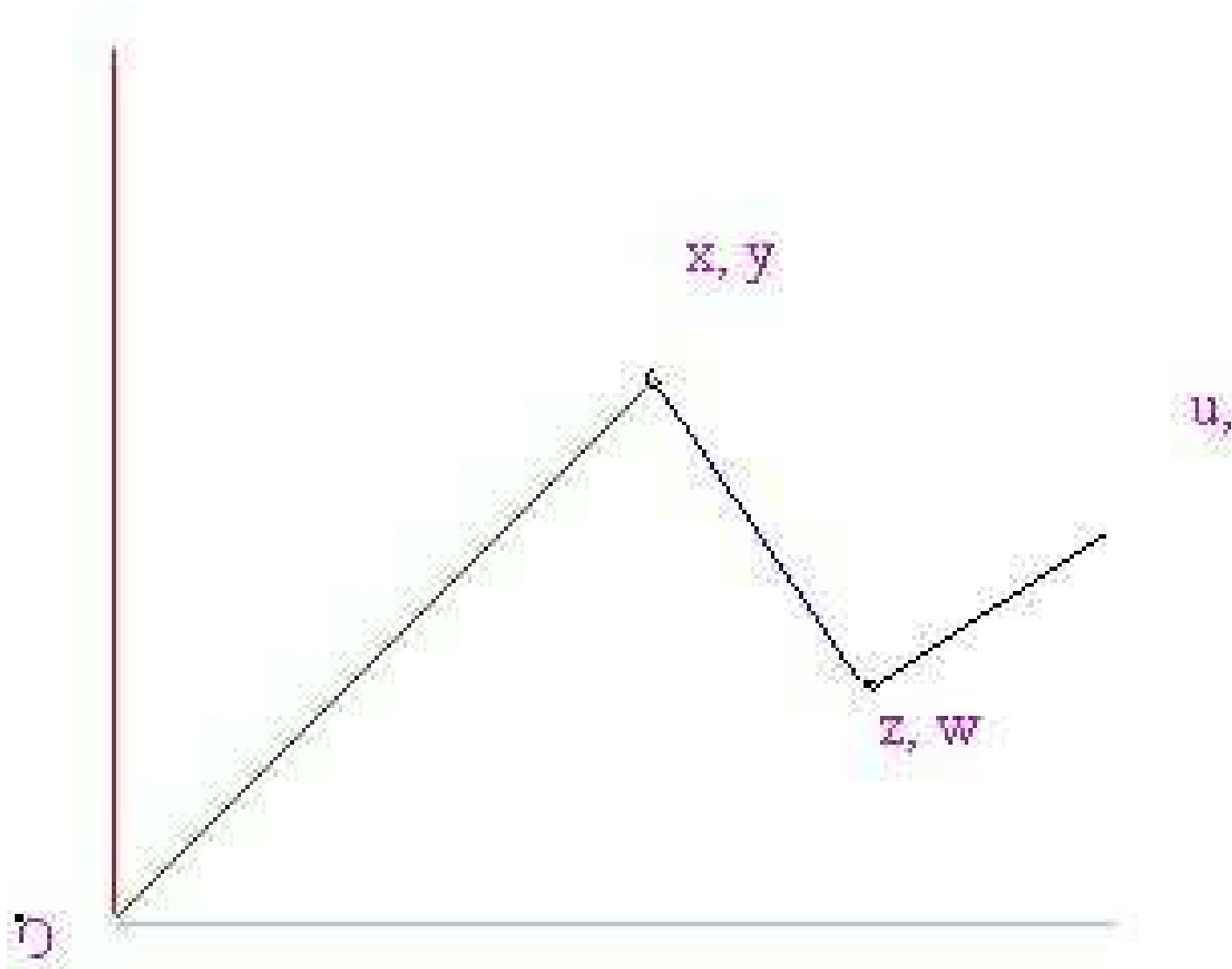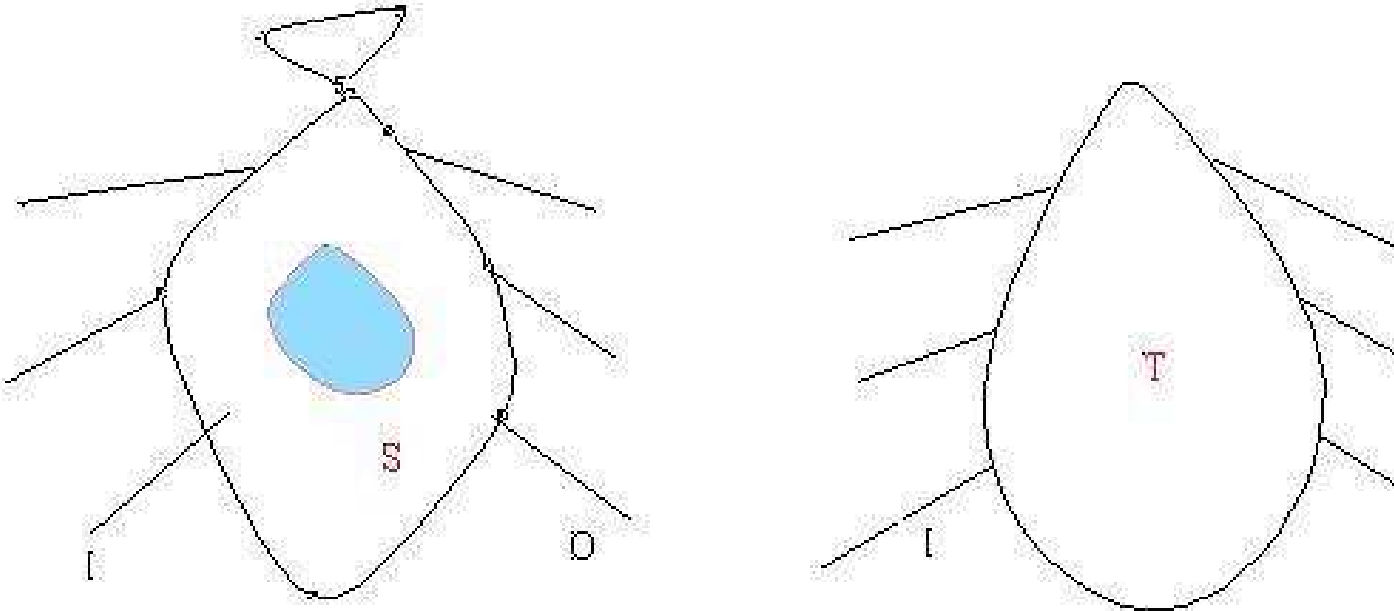

Figure 2: Geometry from Graphs

$$
\begin{aligned}
x^2 + y^2 &= 36 \\
(z - x)^2 + (y - w)^2 &= 16 \\
(u - z)^2 + (v - w)^2 &= 4
\end{aligned}
\tag{16}
$$

So from this simple example two things are clear, the graphs consisting of the neurons and synapses at lattice sites is similarly able to give rise to memory and cognitive states in a manifold, but what is crucial to understand is that as that case is not as simple as the previous case it is quite likely that the structure may quite depend on the nontrivial topologies of the input and random networks of the neuronal configurations. In a sense different topologies of the graphs may give rise to nontrivial manifold and geometric structure. In the diagram below we give a simple example involving the input and output states but with distinct topologies of the neuronal graphs which essentially depend on the neuronal and synaptic configuration connecting those states. In discussing models of memory significantly there are two aspects,

short-term memory (STM), and long-term memory (LTM). Modeling these phenomena has been taken from various possible angles, but noteworthy among them is the importance of dynamical systems and neural networks. What is crucial here is the choice of a proper network model and the memory states may be given by the stability and attractor properties of the networks and choice of a suitable Lyapunov function [51]. On the contrary LTM is mainly concerned with storage patterns associated with modification of synapses, through synaptic plasticity. Our approach here is mainly composed of finding out the geometry of the space in continuation of our preceding analysis and see how memory can be incorporated. In this connection of gaining some geometric intuitions in regards to brain modeling we would like to mention an important result of Principal component analysis [52]. If λ is the input stimuli, and v are the coordinates of the input vector in the state space, a map $\mathcal{K}$ is constructed

such that $\mathcal{K} : \lambda \longmapsto v$. By the process of nonnegative matrix factorisation in which $\mathcal{K}, v$ are positive definite a real image is decomposed into the smaller units by the decomposition of the matrix in lower dimensional eigensubspaces. What we propose here is that the geometry of the surfaces of the corresponding manifold are the states of memory, which are associated with a probability distribution. For example the neighbourhoods of a particular point has a likelihood of being arisen from a set of similar or simultaneous sensory experiences, which are being correlated by the notion of a probabilistic neighbourhood on the manifold. From the conclusions of (14) this distance may be statistical in nature. It is quite already well known from a biological perspective that the central nervous system evolves through natural selection of optimal interactions with the environment. Geometrically this may be expressed by the speculation that the neuronal circuits matches the system of relations among objects in the external world with a many dimensional inner geometry. Recent studies [53]have also showed some possibility of a geometric structure by defining distance function on the cortical areas of the brain. But as far our studies has made us to believe that the input and output states may be related (6) by the probability distribution of the graphs which in turn is being determined by the synaptic weights and neuronal spikes via a differential equation which determines the structure of the manifold in the functional space in the brain. At this point it is quite relevant to look into the state space model which treats the neuronal activities as a noise free dynamical system given by the following set of equations,

$$
\begin{aligned}
x(n+1) &= \varphi(W_i x(n) + W_j u(n)) \qquad (17)\\
y(n) &= \mathcal{H} x(n)
\end{aligned}
$$

where $x(n)$ denote the state (q by 1 vector) of the nonlinear system, $u(n), y(n)$ denote the input (m by 1 vector) and output (p by 1 vector) respectively. $W_i, W_j, \mathcal{H}$ are the q by q matrix, q by (m+1) matrix, p by q matrix respectively. Here φ is a diagonal map defined as $\varphi : \mathcal{R}^q \rightarrow \mathcal{R}^q$ It should be realized that the state space model of networks may be realized in our scheme in (6) by the following identification, $\mathcal{H} \rightarrow \mathcal{P}, W \rightarrow q, \varphi \rightarrow g$ Now it should be realized that this is an approximate and imprecise correspondence which we state at this as a conjecture. What we want to stress at this juncture is that we hope that our model of geometrical picture of the state space may have the potential of a realistic model which will obviously require biological evidences. So if the weight factors can be integrated out or some suitable guesses of those variables may give us a rough idea about the manifold and the possible geometry of the state space. In other words the output memory states are related with a probabilistic measure in the state space in reference to the input states, i.e. there is a distribution of the states on the surfaces of the manifold. But as it should be realized that inputs may form a category in terms of say systems specificity(visual,auditory etc) or functional(colour,shape etc) it is quite important that there would be a variety of overlapping geometries or surfaces from a

geometrical point of view. So we would get a different probability distributions $\mathcal{P}$ for each case. The outputs will also be arranged on the surfaces accordingly. For example N classes of signals may occur in accordance with our model with probabilities $\mathcal{P}_1, \cdots \mathcal{P}_n$ corresponding to distinct graph configurations. This may in turn give rise to distinct and intersecting geometries. But how are they important from a biological perspective? For example suppose the visual field is exposed two sets of stimuli, a red flower and a red rocky mountain in a succession of a large time interval. In the language of tensor network theory these inputs may be correlated in terms of the invariant sets of relationships, but in our framework it is somewhat different. As we have discussed each input gives rise to a different intersecting geometries associated with different probability density. Now again say after long time span suppose the subject is exposed two any one of the stimuli. Then there is a probability that the subject may or may not recall the other stimuli because of the intersecting nature of the associated surfaces. In the figure below (3) we have drawn various surfaces with different colours representing multitude of cognitive states and memory. The surfaces correspond to various stimuli distributed with a definite probability distribution. Intersecting surfaces do show the overlapping nature of the stimuli. The exact description and predictability of the recall phenomena will involve a detailed analysis of the neurodynamical equations and studies of the corresponding geometries evolving out of these. But as we have mentioned before whether a pattern or event will be recalled or not is a statistical phenomena, as long as the memory states do form a short-term memory. So as wee see interpretation or modeling of LTM in our scheme if at all possible is not obvious at this stage. Some recent interesting results are worth mentioning in connection with our investigations. Analysis of BCM theory [43, 54] on the lateral geniculate nucleus for the visual field by the modified hebbian learning process shows surface formation to construct the aspects of learning and image formation. It should be also be stressed that the self organizing maps modeled by [55] do exhibit topological aspects in learning process and memory by the formation of input patterns and synaptic weight formation. To relate with experiments we would like to mention the recent advances in tensor maps by the eigenvectors of the deformation gradient tensor associated with the brain activities which transforms a time evolution of the brain anatomy.

We sketch below some simple examples of finding out geometries of a curve and show that what can the model proposed by us lead to under simpler situations. In parametrizing curves we may state that a point on the curve be given by a parametric function of the following form

$$p(u) \;=\; \sum_{i=0}^{n} p_i f_i(u) \qquad u \in [0,1] \tag{18}$$

where the vectors p_i represent the $n+1$ vertices of a characteristic polygon and $f_i(u)$ are the basis functions. The above parametrization (18) is denoted as a Bezier curve. In drawing this curve the model demands some restrictions on the basis functions and the functions are approximated with some polynomials. In this case a family of functions called Bernstein polynomials satisfy
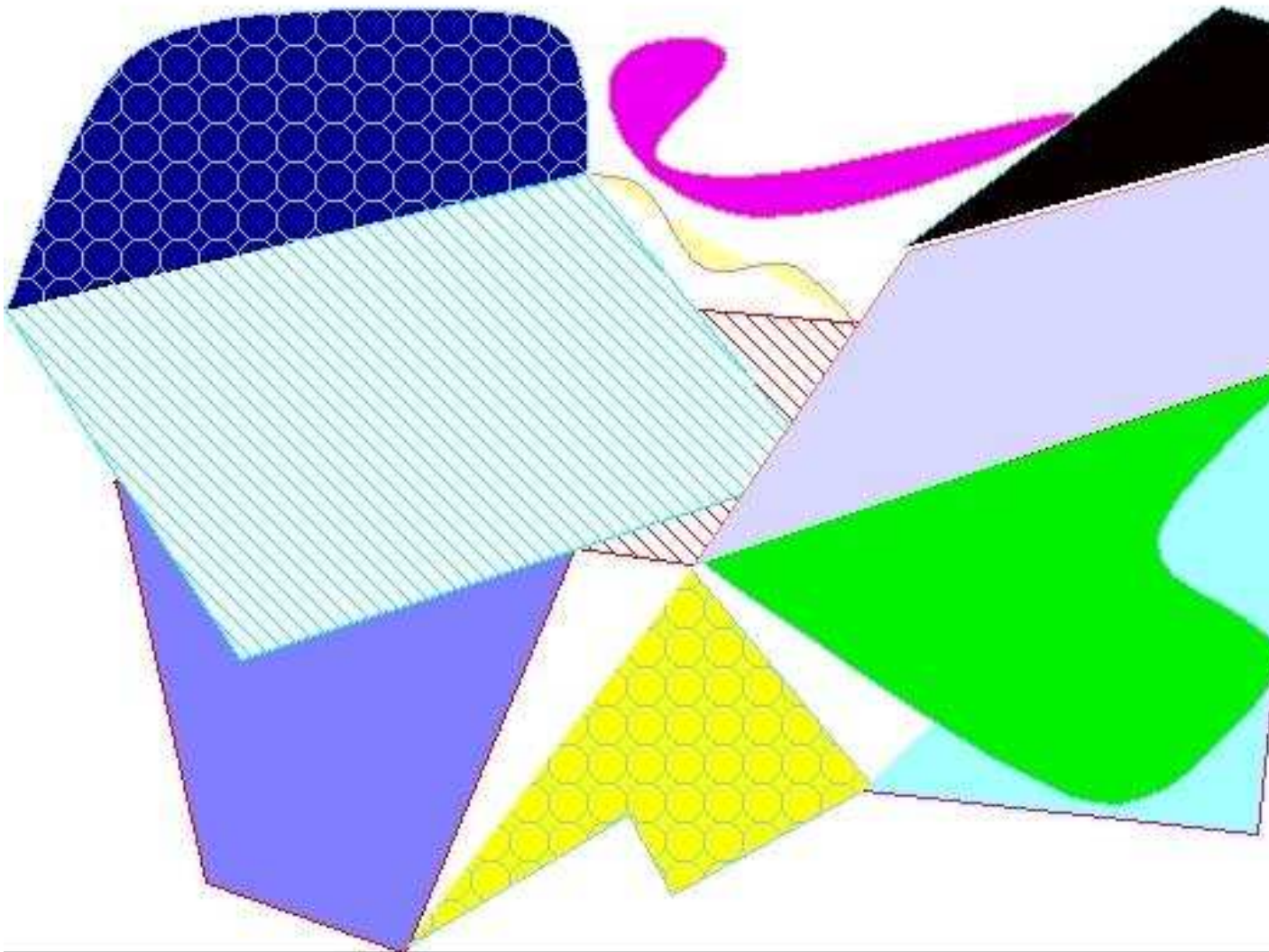
Figure 3: Cognitive States and Algebraic Surfaces

those restrictions and the equation takes the form

$$p(u) \;\; = \;\; \sum_{i=0}^{n} p_i B_{i,n}(u) \qquad u \in [0,1] \tag{19}$$

where the polynomials satisfy

$$B_{i,n}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^i \tag{20}$$

For 3 points n = 2 and the parametrization takes the form

$$p(u) = (1-u)^2 p_0 + 2u(1-u)p_1 + u^2 p_2$$

As it can be understood surfaces can be parametrized along the same lines. So the approach would be to get an idea of the geometry from the dynamical equations and parametrize the geometry by suitable functions for concrete predictability. In our case for example equation (6) for a stationary solution of the probability density $(\mathcal{P})$ equation (9) for simple choices of the weights and neuron spikes do lead a relationships in the input output states of the form [4]

$$\psi^2 + a\psi\lambda + b\psi = \lambda^3 + s\lambda^2 + t\lambda + v \tag{21}$$

Here the a, b, s, t, v are related to the previous coefficients. The equation (21) is termed as the Weierstrass form of an elliptic curve [56]. So we may get a specific geometry $\mathcal{M}_i$ and the memory states for each stimulus are embedded in this geometry with a probability density function $\mathcal{P}_i$.

4   Concluding Remarks

The above analysis clearly indicates that neuronal circuits may give rise to a neuronal geometry by the realization of graphs on a function space. The existence of a distance function and a probabilistic metric tensor unlike some previous analysis [53] is not clear from the discussions presented here. But it is somewhat clear that the geometry is of probabilistic nature. The nature of the paper has been developing some new ideas and proposing its realizations. We hope that this approach may give rise to some interesting conclusions in deeper theoretical and experimental analysis as regards information processing, and theories of cognition, and memory.

Acknowledgements

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[4]We suppress the indices here for typographical convenience

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Retina

CGN

Cortex