Random Recurrent Neural Networks Dynamics.

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Abstract. This paper is a review dealing with the study of large size random recurrent neural networks. The connection weights are varying according to a probability law and it is possible to predict the network dynamics at a macroscopic scale using an averaging principle. After a first introductory section, the section 2 reviews the various models from the points of view of the single neuron dynamics and of the global network dynamics. A summary of notations is presented, which is quite helpful for the sequel. In section 3, mean-field dynamics is developed. The probability distribution characterizing global dynamics is computed. In section 4, some applications of mean-field theory to the prediction of chaotic regime for Analog Formal Random Recurrent Neural Networks (AFRRNN) are displayed. The case of AFRRNN with an homogeneous population of neurons is studied in section 4.1. Then, a two-population model is studied in section 4.2. The occurrence of a cyclo-stationary chaos is displayed using the results of [16]. In section 5, an insight of the application of mean-field theory to IF networks is given using the results of [9].

1 Introduction

Recurrent neural networks were introduced to improve biological plausibility of artificial neural networks such as perceptrons since they display internal dynamics. They are useful to implement associative recall. The first models were endowed with symmetric connexion weights which induced relaxation dynamics and equilibrium states [28]. Asymmetric connexion weights were introduced later on, enabling the observation of complex dynamics and chaotic attractors. The role of chaos in cognitive functions was first discussed by W.Freeman and C.Skarda in seminal papers such as [37]. The practical importance of such dynamics is due to the use of on-line Hebbian learning to store dynamical patterns. For a review see for instance [26].

The nature of the dynamics depends on the connexion weights. When considering large sized neural networks, it is impossible to study the dynamics as a function of the many parameters involved in the network dynamics: parameters defining the state of the neuron such as the Sodium, Potassium conductibility in Hodgkin-Huxley models; parameters defining the structure of the synapses; parameters attached to the environment; external inputs; etc .... One may consider that the connexion weights share few values but this does not allow to study the effect of the variability. Henceforth, one often considers *random* models where the connexion weights form a random sample of a probability distribution. These models are called "*Random Recurrent Neural Networks*" (RRNN).

In this context, the parameters of interest are those defining the probability distribution, i.e. the statistical parameters (introduced as "macroscopic parameters" in the first paper of this review, refereed, from now on, as paper I). Then a study of the dynamics in terms of relevant

dynamical quantities, called *order parameters*[1] can be performed via *"Dynamic Mean-Field Equations"* (DMFE). This terminology and method is inherited from statistical physics and quantum field theory, though is has to be adapted to the present context. Mean-Field Equations were introduced for neural networks by Amari [2] and later on by Crisanti and Sompolinsky [38]. Their results were extended in [12] and proved in a rigorous way in [34]. In [34] the Authors used a **"Large deviation Principle"** (LDP) coming from rigorous statistical mechanics [7]. Simultaneously, mean-field equations were successfully used to predict the dynamics of spiking recurrent neural networks [9,40].

This paper intends to provide a bridge between the detailed computation of the asymptotic regime and the rigorous aspects of MFE theory. We shall introduce the mathematical basis of dynamic mean field theory in sections 2,3, and apply it to several prominent examples. Note however that our approach is different from the standard ones based either on the computation of a functional generating the cumulants, or using an ad hoc approximation replacing the sum of incoming synaptic potentials by a Gaussian random variable. Instead we use large deviations techniques (detailed in the appendix). They have the advantage to be rigorous and they allows to prove convergence results stronger than the usual techniques.

In section 2, the various models are stated from the points of view of the single neuron dynamics and of the global network dynamics. A summary of notations is presented, which is quite helpful for the sequel. In section 3 mean-field dynamics is developed. The probability distribution characterizing global dynamics is computed. The mathematical tools which are used there are detailed (without any proof) in appendix. In section 4, some applications of mean-field theory to the prediction of chaotic regime for analog formal random recurrent neural networks (AFRRNN) are displayed. The dynamical equation of homogeneous AFRRNN, which is studied in paper I, is derived from the random network model in section 4.1. Moreover a two-population model is studied in section 4.2 and the occurrence of a cyclo-stationary chaos is displayed using the results of [16]. In section 5, an insight of the application of mean-field theory to IF networks is given using the results of [9]. The model of this section is a continuous-time model following the authors of the original paper. Hence the theoretical framework of the beginning of the paper has to be enlarged to support this extension of mean-field theory and this work has still to be done. However, we sketch a parallel between the two models to induce further research.

2 Dynamics of Random Recurrent Neural Networks.

2.1 Defining dynamic state variables

The state of an individual neuron i at time t is described by an instantaneous individual variable, the *membrane potential* $u_i(t)$. In stochastic models, such as the ones considered here, all variables (including the $u_i(t)$'s are (real) random variables[2] which takes their values in $\mathbb{R}$. In this section, we consider discrete time dynamics and restrict ourselves to finite time-horizon, i.e. we consider time t as an integer belonging to time interval $\{0, 1, ..., T\}$ where T is finite. Thus an individual state trajectory $u_i \overset{\text{def}}{=} (u_i(t))_{t \in \{0,1,...,T\}}$ takes its value in $\mathcal{F} = \mathbb{R}^{\{0,1,...,T\}}$. Though one is also interested in long-time behaviour and stationary regime (if any), rigorous proofs of convergence of large-size networks only exist for finite time.

We shall study the probability distribution of *trajectories* instead of the probability of an instantaneous state. Actually, the later can be easily obtained from the former. The second order moments of an individual trajectory u_i are its expectation $\mathrm{E}(u_i) \in \mathcal{F}$ and its covariance matrix $\mathrm{Cov}(u_i) \in \mathcal{F} \otimes \mathcal{F}$.

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[1] This terminology comes from statistical physics and was introduced by the physicists L. Landau. Prominent examples of order parameters are the magnetization in the Ising model or the Edwards-Anderson parameter in spin-glasses.

[2] Following the standard physicist's habit, the random variables won't be noted by capital letters.

Our aim is to study the coupled dynamics of N interacting neurons that constitute a neural network. N is the *size* of the neural network. The global state trajectory of $u \stackrel{\text{def}}{=} (u_i)_{i \in \{1...,N\}}$ is a random vector in $\mathcal{F}^N$. The probability distribution[3] of the random vector u, denoted by Q_N depends on N. We shall compute it for various neuron models in this section. We shall first focus to the case of homogeneous neural networks then more realistic cases such as several population networks will be considered.

As it was detailed in paper I, the dynamics of the neuron models that we study here depends on three crucial points.

− how does the neuron activation depend on the membrane potential,
− how do the other neurons contribute to the synaptic potential which summarizes completely the influence of the network onto the target neuron,
− how is the synaptic potential used to update the membrane potential.

We shall now detail these points in the models that are considered further on.

2.2 Spike firing modeling

As introduced in paper I, it is considered that the neuron is active and emits a spike whenever its membrane potential exceeds the *activation threshold*. So the neuron i is active at time t when $u_i(t) \geq \theta$ where θ is the neuron activation threshold. We consider here that θ is constant during the evolution and is the same for all neurons. Actually this hypothesis may be relaxed and random thresholds may be considered but the notation and the framework of dynamical study would be more complicated (see [34]).

For spiking neuron models we define an *activation variable* $x_i(t)$ which is equal to 1 if neuron i emits a spike at time t and to 0 otherwise. Hence we have

$$x_i(t) = f[u_i(t) - \theta] \tag{1}$$

where f, called the *transfer function* of the neuron is here the Heaviside function. Actually, to alleviate notations, we shift u_i of θ and that allows to replace equation (1) by equation

$$x_i(t) = f[u_i(t)] \tag{2}$$

The threshold will be further on taken into account in the updating equation.

Two spiking neuron models are considered here, the *Binary Formal neuron (BF)* which is the original model of Mac Culloch and Pitts [32] and the *Integrate and Fire neuron (IF)* which is generally used nowadays to model dynamics of large spiking neural networks [22].

In these models, the neuron activation takes generally two values: 0 and 1. This is true for most models of neurons. However, it was preferred in a lot of research works to take into account the average firing rate of the model instead of the detailed instant of firing (see section 5.1, paper I). This point of view simplifies the model as it deals with smooth functions easier to handle from a mathematical point of view. In this case, equation (2) is still valid but $x_i(t)$ takes its values in the interval $[0, 1]$ and the transfer function f is a smooth sigmoid function for instance $f(x) = \dfrac{e^x}{1 + e^x}$. Since the activation of the neuron is represented by a real value that varies continuously, the model is called *Analog Formal neuron (AF)*.

AF model is still widely dominant when Artificial Neural Networks are considered for applications since gradient are easy to compute. For biological purpose, it was widely believed that the relevant information was stored in the firing rate; in that case more precise modeling would not be so useful, at least from a functional point of view.

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[3] This term is defined in Appendix, definition 7

The three models are studied in that chapter and we attempt to give a unified presentation of mean-field equation for these three models. Note that, in the sequel, the *state* of the neuron will be the membrane potential $u_i(t)$ and not the activation $x_i(t)$. This is due to the updating definition in the IF model. We shall return to that point later on.

2.3 The synaptic potential of RRNN

The spikes are used to transmit information to other neurons through the synapses. We shall adopt here the very rough, but classical description of the synapse. Namely, the synaptic connexion from neuron j to neuron i is denoted by J_{ij}. It can be positive (excitatory) or negative (inhibitory). Let us denote by $\mathcal{J} = (J_{ij})$ the matrix of synaptic weights. At time 0, the dynamical system is initialized and the synaptic potentials are set to zero[4].

The *synaptic potential* of neuron i of a network of N neurons at time $t + 1$ is expressed[5] as a function of $\mathcal{J}$ and $u(t) \in \mathbb{R}^N$ by

$$v_i(\mathcal{J}, u)(t+1) = \sum_{j=1}^{N} J_{ij} x_j(t) = \sum_{j=1}^{N} J_{ij} f[u_j(t)] \tag{3}$$

(The notation $v_i(\mathcal{J}, u)(t + 1)$ will be explained below).

As discussed in the introduction, we consider here random models where the connexion weights form a random sample of a probability distribution (*"Random Recurrent Neural Networks"*(RRNN)). In that case, the parameters of interest are the statistical parameters defining the probability distribution. A standard example considered in this paper is Gaussian connexion weights. In this case, the statistical parameters are denoted by[6] $\bar{J}$ and J^2 so that $\mathcal{J}$ is a normal random matrix with *independent* components distributed according to the normal law $\mathcal{N}(\frac{\bar{J}}{N}, \frac{J^2}{N})$.

Note that the assumption of independence is crucial in the approach described below. Unfortunately, the more realistic case where correlations between the J_{ij}'s exist (e.g. after Hebbian learning) is, currently, out of reach for all the mean-field methods that we know. We shall first consider Gaussian synaptic synaptic, but we shall extend later on the RRNN model properties to a more general setting where the weights are non Gaussian and depend on the neuron class in a several population model like in [16].

We have already dealt with the dynamical properties of RRNN such as (3) in paper I, considered from the dynamical system point of view, where we fix a realization of the J_{ij}'s and consider the evolution of trajectories of this dynamical system. Then, we have averaged over the J_{ij} distribution in order to get informations about the evolution of averaged quantities. In the present paper we shall start with a complementary point of view. Namely, assume that we *fix the trajectory u_i of each neuron* (resp. we fix the trajectory u of the network). Then, at each time step the variable $\sum_{j=1}^{N} J_{ij} f[u_j(t)]$ is a Gaussian random variable whose probability distribution is induced by the distribution of the J_{ij}'s. Of course, this distribution depends on the trajectory (for example $E[\sum_{j=1}^{N} J_{ij} f[u_j(t)]] = \frac{\bar{J}}{N} \sum_{j=1}^{N} f[u_j(t)]$). To emphasize this dependence we shall denote by $v_i(\mathcal{J}, u) = (v_i(\mathcal{J}, u)(t)) \in \mathcal{F}$ the trajectory of the synaptic potential as in (3).

With this line of reasoning one can show that the $v_i(., u)$ are (conditionally on u) Gaussian identically distributed and *independent* random vectors in $\mathcal{F}$ (see appendix, proposition

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[4] This may for example corresponds to a rest state set to zero without loss of generality (since its corresponds to changing the voltage reference for the membrane potential, see paper I).

[5] See section 5 in paper I.

[6] We use here the same notation as in paper I. Recall that the scaling with $\frac{1}{N}$ allows to have a synaptic potential whose mean and variance are independent of N.

(16). The distribution of v_i is therefore[7] defined by its mean m_u and its covariance matrix c_u (depending on u).

We have

$$m_u(t+1) = \frac{\bar{J}}{N} \sum_{j=1}^{N} f[u_j(t)] \tag{4}$$

and

$$c_u(s+1, t+1) = \frac{J^2}{N} \sum_{j=1}^{N} f[u_j(s)]f[u_j(t)] \tag{5}$$

Notice that these quantities, called *order parameters* in the sequel, are invariant by any permutation of the neuron membrane potentials. Actually, they depend only on the *empirical distribution*[8] μ_u, associated to u.

Definition 1 *The empirical measure is an application from $\mathcal{F}^N$ to $\mathcal{P}(F)$, the set of probability measures on $\mathcal{F}$. It is defined by*

$$\mu_u(A) = \frac{1}{N} \sum_{i=1}^{N} \delta_{u_i}(A) \tag{6}$$

where $\delta_u(A)$ is the Dirac mass on the set A, where $\delta_u(A) = 1$ if u belongs to A and 0 otherwise.

Using this formalism provides an useful way to perform an average over a probability distribution on the trajectories u. For example, the average $\frac{1}{N}\sum_{j=1}^{N} g[u_j(t)]$, where g is some function, writes[9] $\int g(\eta(t))d\mu_u(\eta)$. More generally, assume that we are given a probability distribution μ on the space of trajectories $\mathcal{F}$. Then, one can perform a generic construction of a Gaussian probability on $\mathcal{F}$.

Definition 2 *For any $\mu \in \mathcal{P}(\mathcal{F})$ the Gaussian probability distribution g_μ on R^T, with moments m_μ and c_μ, is defined by :*

$$\begin{cases} m_\mu(t+1) = \bar{J} \int f[\eta(t)]d\mu_u(\eta) \\ c_\mu(s+1, t+1)] = J^2 \int f[\eta(s)]f[\eta(t)]d\mu_u(\eta) \end{cases} \tag{7}$$

Then, it is easy to reformulate the previous computation as:

Proposition 1 *The common probability distribution of the individual synaptic potential trajectories $v_i(., u)$ is the normal distribution g_{μ_u} where μ_u is the empirical distribution of the network potential trajectory u.*

This framework is useful to compute the large-size limit of the common probability distribution of the potential trajectories.

2.4 Dynamical models of the membrane potential

We shall now detail the updating rule of the membrane potential. Various neural dynamics have been detailed in paper I. We focus here on Analog Formal (AF), Binary Formal (BF), and Integrate and Fire neuron (IF).

In any case, the network is initialized with independent identically distributed membrane potential according to a probability distribution $m_0 \in \mathcal{P}(\mathbb{R})$. It is useful, on technical grounds, to add to each neuron a small amount of noise. Thus we introduce for each neuron i, a sequence

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[7] This distribution actually does not depend on i since the v_i's are identically distributed.

[8] This concept is introduced within more details in the appendix, definition (18)

[9] Note that the integration variable η corresponds to a *trajectory* of the network, and that $\eta(t)$ corresponds to the *state of the network* at time t (i.e. $\eta_i(t)$ is the state of neuron i at time t).

$(w_i)(t))_{t\in\{1,...,T\}}$ of i.i.d. centered Gaussian variables of variance σ^2. This sequence is called the *synaptic noise* of neuron i. The synaptic noise plays an important part in the mathematical proof but the order parameter σ is as small as necessary. So this model is not very restrictive. The synaptic noise is added to the synaptic potential. On biological grounds it may account for different effects such as the diffusion of neurotransmitters involved in the synaptic transmission, the degrees of freedom neglected by the model, external perturbations, etc ... Though it is not evident that the "real noise" is Brownian, using this kind of perturbations has the advantage of providing a tractable model where standard theorems in the theory of stochastic processes or methods in non equilibrium statistical physics (e.g. Fokker-Planck equations) can be applied. In some papers, the synaptic noise is called *thermal noise or annealed noise* by comparison with the random variables $\mathcal{J} = (J_{ij})$, which are called *quenched variables* as they are fixed, once for all, and do not change with time (we do not consider learning in this paper).

The formal neuron updates its membrane potential according to

$$u_i(t+1) = v_i(t+1) + w_i(t+1) - \theta \tag{8}$$

IF neuron takes into account its present membrane potential while updating. Its evolution equation is

$$u_i(t+1) = \varphi[u_i(t) + \theta] + v_i(t+1) + w_i(t+1) - \theta \tag{9}$$

where

- φ is defined by

$$\varphi(u) = \begin{cases} \gamma u \text{ if } \frac{\vartheta}{\gamma} < u < \theta \\ \vartheta \text{ else} \end{cases} \tag{10}$$

- $\gamma \in]0,1[$ is the *leak* (damping coefficient).
- ϑ is the reset potential and $\vartheta < 0 < \theta$

The following table summarizes the main properties of the three models we investigate:

Transfer function	Heaviside	sigmoidal
Formal model	BF	AF
Integrate and Fire	IF	

Assume for a moment that we remove the neural coupling, then the individual neuron state trajectories are independent, identically distributed, random vectors in $\mathcal{F}$ (whose randomness is induced by the Brownian noise). The corresponding dynamics is called the *free dynamics*. Let us denote by P the common distribution of the neurons trajectory in the uncoupled case. The probability distribution of the corresponding neural network trajectory is therefore $P^{\otimes N}$.

In the case of formal neurons, the free dynamics equation is

$$u_i(0) \sim m_0, \quad u_i(t+1) = w_i(t+1) - \theta \tag{11}$$

where $\sim$ means "distributed according to". So $P = m_0 \otimes \mathcal{N}(-\theta, \sigma^2)^{\otimes T}$. In the case of IF neurons P is not explicit. It is the image of $m_0 \otimes \mathcal{N}(-\theta, \sigma^2)^{\otimes T}$ by the diffusive [10] dynamics

$$u_i(0) \sim m_0, \quad u_i(t+1) = \varphi[u_i(t) + \theta] + w_i(t+1) - \theta \tag{12}$$

When coupling the neurons, the trajectory of the system of neurons is still a random vector. Its probability distribution, denoted by Q_N, has a density with respect to $P^{\otimes N}$ that can be explicitly computed. This is the main topic of the next subsection.

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[10] This is indeed a discrete time stochastic difference equations with drift $\varphi[u_i(t)+\theta]-\theta$ and diffusion $w_i(t+1)$. Incidentally, we shall come back to stochastic differential equations for neural networks in the section 5 and we shall consider the related Fokker-Planck equation.

2.5 Computation of the probability distribution of network trajectories.

This section is devoted to the computation of the probability distribution Q_N. The result shows that the density of Q_N with respect to the free dynamics probability $P^{\otimes N}$ depends on the trajectory variable u only through the empirical measure μ_u. To achieve this computation we shall use a key result of stochastic process theory, the Girsanov theorem [23, 25], which gives the density of the new distribution of a diffusion when the drift is changed. Actually, since the time set is finite, the version of Girsanov theorem that we use is different from the original one [25] and may be recovered by elementary Gaussian computation. Its derivation is detailed in the appendix in theorem 19. A similar result may be obtained for continuous time dynamics using the classical Girsanov theorem (see [7]).

Let us state the finite-time Girsanov theorem

Theorem 2 *Let m_0 a probability measure on $\mathbb{R}^d$ and let $\mathcal{N}(\alpha, K)$ be a Gaussian regular probability on $\mathbb{R}^d$ with mean α and covariance matrix K. Let T be a positive integer and $\mathcal{E}_T = (\mathbb{R}^d)^{\{0,\dots,T\}}$ be the space of finite time trajectories in $\mathbb{R}^d$. Let w be a Gaussian random vector in $\mathcal{E}_T$ with distribution $m_0 \otimes \mathcal{N}(\alpha, K)^T$. Let ϕ and ψ be two measurable applications of $\mathbb{R}^d$ into $\mathbb{R}^d$. Then we define the random vectors x and y in $\mathcal{E}$ by:*

$$\begin{cases} x_0 = w_0 \\ x(t+1) = \phi[x(t)] + w(t+1) \end{cases} \tag{13}$$

$$\begin{cases} y_0 = w_0 \\ y(t+1) = \psi[y(t)] + w(t+1) \end{cases} \tag{14}$$

Let P and Q be the respective probability distributions on $\mathcal{E}$ of x and y, then we have:

$$\frac{dQ}{dP}(\eta) = \tag{15}$$

$$\exp \sum_{t=0}^{T-1} \left\{ \begin{array}{l} -\frac{1}{2}\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\psi[(\eta(t)] - \phi[(\eta(t)]\} \\ +\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\eta(t+1) - \alpha - \phi[\eta(t)]\} \end{array} \right\}$$

namely Q is <u>absolutely continuous</u> with respect to P.

We shall use this theorem to prove the following:

Theorem 3 *The density of the distribution of the network membrane potential Q_N with respect to $P^{\otimes N}$ is given by*

$$\frac{dQ_N}{dP^{\otimes N}}(u) = \exp N\Gamma(\mu_u) \tag{16}$$

where the functional Γ is defined on $\mathcal{P}(\mathcal{F})$ by:

$$\Gamma(\mu) = \int \log \left\{ \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2}\xi(t+1)^2 + \Phi_{t+1}(\eta)\xi(t+1) \right] dg_\mu(\xi) \right\} d\mu(\eta) \tag{17}$$

with:

- *for AF and BF models: $\Phi_{t+1}(\eta) = \eta(t+1) + \theta$*
- *IF model: $\Phi_{t+1}(\eta) = \eta(t+1) + \theta - \varphi[\eta(t) + \theta]$*

Remark 1 *Let us recall that the Gaussian measure g_μ has been defined previously (Definition 2)*

PROOF OF THEOREM: Call $Q_N(\mathcal{J})$ the conditional distribution of the network state trajectory given $\mathcal{J}$, the matrix of synaptic weights. We shall apply the finite-time Girsanov theorem 2 to express $\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}$. To apply the theorem we notice that

– The difference of the two drift terms $\psi[\eta(t) - \phi[\eta(t)]$ of the theorem is here the synaptic potentials $(v_i(t))$. The synaptic potentials v_i are functions of the u_i's according to (3)

$$v_i(\mathcal{J}, u)(t+1) = \sum_{j=1}^{N} J_{ij} f[u_j(t)]$$

– The expression of the synaptic noise $(w_i(t+1)$, as a function of the state trajectory u_i in the free dynamics, is given by $\Phi_{t+1}(u_i)$. The explicit form of Φ depends on the neuron model (formal or IF).

We have so

$$\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}(u) = \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \sum_{i=1}^{N} \left[-\frac{1}{2} v_i(\mathcal{J}, u)(t+1)^2 + v_i(\mathcal{J}, u)(t+1)\Phi_{t+1}(u_i) \right] \quad (18)$$

$$\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}(u) = \prod_{i=1}^{N} \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2} v_i(\mathcal{J}, u)(t+1)^2 + v_i(\mathcal{J}, u)(t+1)\Phi_{t+1}(u_i) \right] \quad (19)$$

We have thus obtained the probability distribution of the membrane potentials in the coupled neurons models, but *for a fixed realization of the J_{ij}'s* (conditional probability).

Let us now consider the probability of the quenched variables $\mathcal{J} = (J_{ij})$. We observed previously when we introduced the synaptic potential model that under the configuration distribution of $\mathcal{J}$, the random vectors $v_i(\mathcal{J}, u)$ are independent identically distributed according to the normal distribution g_{μ_u}. To compute the density of Q_N with respect to $P^{\otimes N}$ one has thus to average the conditional density $\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}$ over the distribution of $\mathcal{J}$. *Since the J_{ij}'s are independent, the integration separates into products* and one gets from (19) the following

$$\frac{dQ_N}{dP^{\otimes N}}(u) = \exp \sum_{i=1}^{N} \log \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2} \xi(t+1)^2 + \Phi_{t+1}(u_i)\xi(t+1) \right] dg_{\mu_u}(\xi)$$

The sum over i is equivalent to an integration over the empirical measure μ_u (6), so we have

$$\frac{dQ_N}{dP^{\otimes N}}(u) = \exp N \int \log \left\{ \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2} \xi(t+1)^2 + \Phi_{t+1}(\eta)\xi(t+1) \right] dg_{\mu_u}(\xi) \right\} d\mu_u(\eta)$$

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Remark. These equations reminds the generating functional approach derived e.g. by Sompolinsky & al. [38,13] or Molgedey & al [31] allowing to compute the moments of the $u_i(t)$'s. However, the present approach provides a stronger result. While the generating functional method deals with weak convergence (convergence of generating function) the method developed here allows to obtain directly the probability distribution of the $u_i(t)$'s. Moreover, by using large deviations techniques one is able to establish almost-sure convergence results (valid for only one typical sample).

Let us now state an important corollary of this theorem.

Corollary 4 *The empirical measure μ_u is a random measure governed by Q_N. It has a density with respect to the distribution of the empirical measure of the free model, (that is governed by $P^{\otimes N}$), given by:*

$$\mu \in \mathcal{P}(\mathcal{F}) \rightarrow \exp N\Gamma(\mu)$$

2.6 Summary of notations

Let us recall the notations of this section. They will be extensively used in the following sections:

Notation	Interpretation
$i \in \{1, ..., N\}$	individual neuron in a N neuron population
$t \in \{0, ..., T\}$	time label of the discrete time dynamics at horizon T
$u_i(t)$	membrane potential of neuron i at time t
$x_i(t)$	activation state of neuron i at time t
$v_i(t)$	synaptic potential of neuron i at time t
$w_i(t)$	synaptic summation noise of neuron i at time t
$u_i \in \mathcal{F}$	membrane potential trajectory of neuron i from time 0 to time T
$u \in \mathcal{F}^N$	network membrane potentials trajectory (network trajectory) from time 0 to time T
$x_i \in \mathcal{E}$	activation state trajectory of neuron i from time 0 to time T
$x \in \mathcal{F}^N$	network activation state trajectory from time 0 to time T
θ	common firing threshold of individual neurons
σ	common standard deviation of the synaptic noise
λ	leak current factor for Integrate and fire (IF) neuron model
f	neuron transfer function converting membrane potential into activation state
J_{ij}	synaptic weight from neuron j to neuron i (real random variable)
$\mathcal{J} = (J_{ij})$	synaptic weight matrix (random $N \times N$ matrix)
$\frac{\bar{J}}{N}$	expectation of synaptic weights J_{ij}
$\frac{J^2}{N}$	variance of synaptic weights J_{ij}
$\mu \in \mathcal{P}(\mathcal{F})$	generic probability distribution of individual membrane potential trajectory
$\eta \in \mathcal{F}$	random vector which takes its values in $\mathcal{F}$ under probability distribution μ
$P \in \mathcal{P}(\mathcal{F})$	probability distribution of individual membrane potential trajectory for free dynamics
$g_\mu \in \mathcal{P}(\mathcal{F})$	synaptic potential distribution obtained from $\mu \in \mathcal{P}(F)$ through central limit approximation
$Q_N \in \mathcal{P}(\mathcal{F}^N)$	probability distribution of network membrane potential trajectory u

3 The mean-field dynamics

3.1 Introduction to mean-field theory

The aim of this section is to describe the evolution of a typical neuron in the limit of large size networks. This is done by summarizing, in a single term, the effect of the interactions of this neuron with the other neurons of the network. A mean-field theory provides evolution equations of the type of free dynamics equation, that are involving single neuron dynamics, but where an *effective interaction term* remains. This term, or "mean-field", is properly the average effect of all the interaction of other neurons with the neuron of interest. So the mean field dynamics is intermediate between the *detailed* dynamics, which takes into account all the detailed interactions between neurons, and the *free* dynamics, which neglects all interactions.

To derive mean-field equations in a direct way, one can replace $v_i(t)$ by an approximation which depends only on the statistical distribution of the $u_j(t)$'s. This approximation takes advantage of the large number of synapses J_{ij} to postulate the vanishing of individual correlations between neurons or between neurons and configuration variables. This is the hypothesis of "local chaos" of Amari ([1],[2]), or of "vanishing correlations" which is usually invoked to support mean-field equations. In the present context, it can be stated as follows.

In the large size limit, the u_j's are asymptotically independent, they are also independent from the configuration parameters

This approach is very similar to the Boltzmann's "molecular chaos hypothesis"[11] introduced by Boltzmann (1872), tailored to neural networks dynamics.

From the central limit theorem we are then allowed to state that the random variable $\zeta(t+1) = \sum_{j=1}^{N} J_{ij} f(u_j(t))$ is a large sum of approximatively independent identically distributed variable, and thus that it has approximatively a Gaussian distribution. Thus, we just have to derive its first and second order moment from the common probability distribution of $u_i = (u_i(t))$ to know completely the distribution of ζ. Henceforth, from a probability distribution on $\mathcal{F}$ which is supposed to be the common probability distribution of the u_i's, we are able to derive the distribution of ζ and then the distribution of the resulting potential trajectory and the state trajectory of a generic vector of the network.

The assumption on which the mean-field approximation is based may look entirely wrong at first glance. However, in the present context it gives exactly the same results as more elaborated methods such as the generating functional method, or the large deviations approach developed below. Moreover, it is supported by the "propagation of chaos" result proved in section 3.4. Note however that in models with correlated interactions J_{ij} (such as spin-glasses, where $J_{ij} = J_{ji}$) the "local chaos" hypothesis leads to wrong results (at low temperature) while generating functional methods [39,15,14] and large deviations techniques [7] can still be used.

3.2 Mean-field propagation operator and mean-field equation

We now define an evolution operator L, on the set $\mathcal{P}(F)$ of probability distributions on $\mathcal{F}$, that we call the *mean-field propagation* operator (or mean-field propagator). Let $\mu \in \mathcal{P}(\mathcal{F})$ be a probability measure on $\mathcal{F}$. Let us compute the moments of

$$\forall t \in \{0, 1, ..., T-1\}, \zeta(t+1) = \sum_{j=1}^{N} J_{ij} f[u_j(t)]$$

where the u_j's are independent identically distributed random vectors with probability distribution μ. They are also independent from the configuration parameters J_{ij}.

Since $E[J_{ij}] = \frac{\bar{J}}{N}$ and $\text{Var}[J_{ij}] = \frac{J^2}{N}$, we have

$$\begin{cases} E[\zeta(t+1)] = \bar{J} \int_{\mathcal{F}} f[\eta(t)] d\mu(\eta) \\ \text{Cov}[\zeta(s+1), \zeta(t+1)] = J^2 \int_{\mathcal{E}} f[\eta(s)] f[\eta(t)] d\mu(\eta) \end{cases} \tag{20}$$

Notice that the expression of the covariance is asymptotic since the sum of squares of expectation of the synaptic weights may be neglected. So ζ is a Gaussian random vector in $\mathcal{F}$ with probability distribution g_μ (see definition 2).

Definition 3 *Let μ a probability distribution on $\mathcal{F}$ such that the distribution of the first component is m_0. Let u, w, v be three independent random vectors with the following distributions*

- *the distribution of u is μ,*
- *the distribution of w is $\mathcal{N}(0, \sigma^2 I_T)$,*
- *the distribution of v is g_μ*

---

[11] The word "chaos" is somehow confusing here, especially because we also dealt with deterministic chaos in the paper I. Actually, "deterministic chaos" and the related exponential correlation decay can be invoked, in statistical physics, to obtain (deterministic) equations for the mean value and Gaussian fluctuations of relevant observables. In the present case the basic reason that makes the mean-field approach "works" is however different. This is the mere fact that the model is fully connected and that the J_{ij}'s are independent and vanishing in the limit $N \to \infty$. This is standard result in statistical physics models such as the Curie-Weiss model but obtaining this for the trajectories of a dynamical model with quenched disorder requires more elaborated techniques.

Then $L(\mu)$ is the probability distribution on $\mathcal{F}$ of the random vector ϑ which is defined by

$$\begin{cases} \vartheta(0) = u(0) \\ \vartheta(t+1) = v(t+1) + w(t+1) - \theta \end{cases} \tag{21}$$

for the formal neuron models (BF and AF), and by

$$\begin{cases} \vartheta(0) = u(0) \\ \vartheta(t+1) = \varphi[u(t) + \theta] + v(t+1) + w(t+1) - \theta \end{cases} \tag{22}$$

for the IF neuron model. The operator L which is defined on $\mathcal{P}(\mathcal{F})$ is called the <u>mean-field propagation operator</u>.

Definition 4 *The following equation on $\mu \in \mathcal{P}(\mathcal{F})$*

$$L(\mu) = \mu \tag{23}$$

is called the **mean-field equation (MFE)**

Remark 2 *The mean-field equation is the achievement of mean-field approach. To determine the distribution of an individual trajectory, we suppose that this distribution governs the inter-action of all the units onto the selected one. The resulting distribution of the selected unit has to be the same distribution than the generic distribution. This is summarized in the mean-field equation*

$$L(\mu) = \mu$$

Equations (21) (resp. (22) for the IF model) with the specification of the probability distributions define the *mean-field dynamics*. Actually, the distribution $L(\mu)$ is just the convolution of the probability distributions P and the Gaussian distribution g_μ. More precisely, if we apply the discrete time Girsanov theorem 19 of the appendix, we have:

Theorem 5 *$L(\mu)$ is absolutely continuous with respect to P and its density is given by*

$$\frac{dL(\mu)}{dP}(\eta) = \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2}\xi(t+1)^2 + \varPhi_{t+1}(\eta)\xi(t+1) \right] dg_\mu(\xi) \tag{24}$$

PROOF : The proof is essentially a simplified version of the application of the finite-time Girsanov theorem which was used to prove theorem (3). The conditioning is done here with respect to v which is the difference between the drift terms of the free dynamics and of the mean-field dynamics. ∎

Remark 3 *We have to notice for further use that*

$$\varGamma(\mu) = \int \log \frac{dL(\mu)}{dP}(\eta) d\mu(\eta) \tag{25}$$

In all the cases, for $0 < t < T$ the projection of the distributions $\varGamma(\mu)$ and $L(\mu)$ on the $t+1$ first time steps just depends on the projection of μ on the t first instants. Since the projection of μ on the initial instant is always m_0, the projection of $L(\mu)$ on the two first instants $\{0, 1\}$ depend only on m_0 and similarly, the projection of $L^t(\mu)$ on the $t+1$ first instants $\{0, 1, ..., t\}$ depends only on m_0. Eventually $\mu_T = L^T(\mu) = L^T(P)$ depends only on m_0 and it is the only fixed point of the mean-field propagation operator L.

So we have shown the following

Theorem 6 *The probability measure $\mu_T = L^T(P)$ is the only solution of the mean-field equation with initial condition m_0.*

3.3 Large Deviation Principle for RRNN mean-field theory

In this section, we fully use the computation results of the previous section to show the rigorous foundations of mean-field theory for RRNN. The approach is the following:

(a) The empirical measure μ_u of the network dynamics satisfies a *large deviation principle (LDP)* under $P^{\otimes N}$ with a good rate function $\mu \in \mathcal{P}(\mathcal{F}) \to I(\mu, P) \in \mathbb{R}^+$, the relative entropy between μ and P. Actually, when the size of the network tends to infinity, the empirical measure converges in distribution exponentially fast towards P. The definition of LDP and its consequences are outlined in the appendix in definition 3.3. Sanov theorem is stated in appendix, theorem 24.

(b) According to corollary 4, the density of the new distribution of μ_u with respect to the original distribution when we switch from $P^{\otimes N}$, that governs the free dynamics, to Q_N, that governs the RRNN dynamics is $\exp N\Gamma(\mu)$.

(c) Combining (a) and (b), one obtains that under Q_N, the sequence μ_u satisfies a LDP with the good rate function

$$H(\mu) = I(\mu, P) - \Gamma(\mu) \tag{26}$$

This kind of result is used in statistical physics under the name of *Gibbs variational principle* [21]. The functional H is called, in statistical physics, a *thermodynamic potential* (e.g. free energy or Gibbs potential). Notice that the classical statistical mechanics framework is relative to equilibrium probability distributions on the space of microscopic states. It is applied here to trajectories. For that reason, this approach is called the *dynamic mean-field theory* [38]. It is quite technical to support it rigorously. One has to show that H is lower semi-continuous and is a good rate function (see Varadhan's theorem 23 of the appendix). This kind of proof is rather technical. To reduce the size of the paper we admit the following result (see [7] for a general approach and [34] for the proof for AFRRNN model)

Theorem 7 *Under the respective distributions Q_N the family of empirical measures (μ_N) of $\mathcal{P}(\mathcal{F})$ satisfies a full large deviation principle with a good rate function H given by (26).*

(d) It is clear from remark 3 that $H(\mu_T) = 0$ where μ_T is is the unique solution of MFE with initial condition m_0, so it is the fixed point of L. Thus μ_T is a minimum of H.

The basic computation is the following: first we apply the definition 19 of the relative entropy that is given in the appendix

$$I(\mu_T, P) = \int \log \frac{d\mu_T}{dP}(\eta) d\mu_T(\eta)$$

Since μ_T is the solution of MFE, we have

$$\frac{d\mu_T}{dP}(\eta) = \frac{dL(\mu_T)}{dP}(\eta)$$

then we apply the previous remark 3 which states

$$\Gamma(\mu_T) = \int \log \frac{dL(\mu_T)}{dP}(\eta) d\mu_T(\eta)$$

to check

$$I(\mu_T, P) = \Gamma(\mu_T) \Rightarrow H(\mu_T) = 0$$

(e) To obtain the exponential convergence of the sequence of empirical measures μ_u under Q_N when $N \to \infty$, one has eventually to show that $H(\mu) = 0 \Rightarrow \mu = \mu_T$. This point is technical too. It is proved in a similar still more general framework (continuous time) in [7] using a Taylor expansion. The same method is and applied to show the uniqueness for AFRRNN model in [34].

Thus, we have the main result of that section:

Theorem 8 *When the size N of the network goes to infinity, the sequence of empirical measures (μ_u) converges in probability exponentially fast towards μ_T which is the unique solution of the mean-field equation $L(\mu) = \mu$*

3.4 Main results of RRNN mean-field theory

First notice that theorem 8 may be extended to RRNN with fast decreasing connection weights distribution. More precisely, assume that the common distribution ν_N of the connexion weights satisfies the following:

Hypothesis 9 (H) *For all N, the common probability law ν_N of the connexion weights satisfies*

$$
\begin{array}{ll}
(i) & \int w d\nu_N(w) = \frac{\bar{J}}{N} \\
(ii) & \int w^2 d\nu_N(w) = \frac{\bar{J}^2}{N} + \frac{\bar{J}^2}{N^2} \\
(iii) & \exists a > 0, \exists D > 0 \ such \ that \ \int \exp(aNw^2) d\nu_N(w) \le D
\end{array}
$$

then the family (ν_N) is said to satisfy hypothesis (H).

Then, from theorem 8 two important results can be deduced rigorously. The first one is a "propagation of chaos" result which supports the basic intuition of mean field theory about the asymptotic independence of finite subsets of individuals when the population size grows to infinity.

Theorem 10 *Let k be a positive integer and $(f_i)_{i \in \{1,...,k\}}$ be k continuous bounded functions on $\mathcal{F}$, when the size N of the network goes to infinity, then*

$$
\int \prod_{i=1}^{k} f_i(u_i) dQ_N(u) \rightarrow \prod_{i=1}^{k} \int f_i(\eta) d\mu_0(\eta)
$$

PROOF : The idea of the proof is due to Snitzman [41].

First, a straightforward consequence of theorem 8 is that when we apply the sequence of random measures (μ_N) to the test function F on $\mathcal{P}(\mathcal{F})$ defined by $F(\mu) = \prod_{i=1}^{k} \int f_i(u_i) d\mu(u)$ we get the convergence of

$$
\lim_{N \to \infty} \int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j) \right] dQ_N(u) = \prod_{i=1}^{k} \int f_i(\eta) d\mu_0(\eta)
$$

Thus it remains to compare $\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j) \right] dQ_N(u)$ and $\int \prod_{i=1}^{k} f_i(u_i) dQ_N(u)$ From the symmetry property of Q_N, it is clear that for any subset $\{j_1, ..., j_k\}$ of k neurons among N, we have

$$
\int \prod_{i=1}^{k} f_i(u_{j_i}) dQ_N(u) = \int \prod_{i=1}^{k} f_i(u_i) dQ_N(u)
$$

If we develop $\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j) \right] dQ_N(u)$, we get

$$
\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j) \right] dQ_N(u) = \frac{1}{N^k} \sum_{\{j_1,...,j_k\}} \int \prod_{i=1}^{k} f_i(u_{j_i}) dQ_N(u) \qquad (27)
$$

The average sum in (27) is here over all applications of $\{1, ..., k\}$ in $\{1, ..., N\}$. And the equality is proved if we replace it by the average over all injections of $\{1, ..., k\}$ in $\{1, ..., N\}$, since the terms are all equal for injections. But when N goes to infinity the proportion of injections which is $\frac{N!}{(N-k)!N^k}$ goes to 1 and thus the contributions of repeated k-uple is negligible when N is large. Therefore

$$
\lim_{N \to \infty} \left[\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j) \right] dQ_N(u) - \int \prod_{i=1}^{k} f_i(u_i) dQ_N(u) \right] = 0
$$

■

Still, this propagation of chaos result is valid when the expectation of the test function is taken with respect to the connection distribution. Thus, it doesn't say anything precise about the observation relative to a *single* large-sized network.

Actually, since exponentially fast convergence in probability implies almost-sure convergence form Borel-Cantelli lemma, we are able to infer the following statement from theorem 8. Recall that we note (as in the proof of theorem 3) $Q_N(\mathcal{J})$ the conditional distribution of the network state trajectory given $\mathcal{J}$ the system of synaptic weights and we define $\mu_N(u) = \frac{1}{N} \sum_{i=1}^{N} \delta_{u_i}$ for the empirical measure on $\mathcal{F}$ which is associated to a network trajectory $u \in \mathcal{F}^N$.

Theorem 11 *Let F be a bounded continuous functional on $\mathcal{P}(\mathcal{F})$, we have almost surely in $\mathcal{J}$*

$$\lim_{N \to \infty} \int F[\mu_N(u)] dQ_N(\mathcal{J})(u) = F(\mu_T)$$

Note that we cannot use this theorem to infer a "quenched" propagation of chaos result similar to theorem 10, which was an "annealed" propagation of chaos result (i.e. averaged over the connection weight distribution). This is not possible because, for a given network configuration $\mathcal{J}$, $Q_N(\mathcal{J})$ is no more symmetric with respect to the individual neurons. Nevertheless, we obtain the following crucial result, applying theorem 11 to the case where F is the linear form $F(\mu) = \int f d\mu$

Theorem 12 *Let f be a bounded continuous function on $\mathcal{F}$, we have almost surely in $\mathcal{J}$*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \int f(u_i) dQ_N(\mathcal{J})(u) = \int f(\eta) d\mu_T(\eta)$$

The consequences of these results are developed in the next section.

4 Mean-field dynamics for analog networks

We are interested in the stationary dynamics of large random recurrent neural networks. Moreover since we want to study the meaning of oscillations and of (deterministic) chaos observed in the finite sized models (see paper I), the regime of low noise is specially interesting since the oscillations are practically canceled if the noise is too strong. For these reasons, we cannot be practically satisfied by obtaining the limit μ_T of the empirical measures. So we shall extract from μ_T dynamical informations on the asymptotics of the network trajectories. Notice that the distribution of the connexion weight distribution is not necessarily Gaussian as long as it satisfies hypothesis (H:9).

4.1 Mean-field dynamics of homogeneous networks

4.1.1 General mean-field equations for moments

Recall that in section 2 of this chapter (definition 2) we defined for any probability measure $\mu \in \mathcal{P}(\mathcal{F})$ the two first moments of μ, m_μ and c_μ. Let us recall these notations:

$$\begin{cases} m_\mu(t+1) = \bar{J} \int f[\eta(t)] d\mu(\eta) \\ c_\mu(s+1, t+1) = J^2 \int f[\eta(s)] f[\eta(t)] d\mu(\eta) \\ q_\mu(t+1) = c_\mu(t+1, t+1) \end{cases}$$

where f is the sigmoid function $f(x) = \dfrac{e^x}{1 + e^x}$

In this section, in order to alleviate notations, we note m, c, q instead of $m_{\mu_T}, c_{\mu_T}, q_{\mu_T}$ where μ_T is the asymptotic probability that was shown to be a fixed point of the mean-field evolution operator L in last section. By expressing that μ_T is a fixed point of L, we shall produce some evolution autonomous dynamics on the moments m, c, q.

More precisely we have from the definition of L (see definition 3 in section 3) that the law of $\eta(t)$ under μ_T is a Gaussian law of mean $m(t) - \theta$ and of variance $q(t) + \sigma^2$ (see equations (20) and (21)). So we have

$$\begin{cases} m(t+1) = \bar{J} \int f[\sqrt{q(t)+\sigma^2}\xi + m(t) - \theta]d\gamma(\xi) \\ q(t+1) = J^2 \int f[\sqrt{q(t)+\sigma^2}\xi + m(t) - \theta]^2 d\gamma(\xi) \end{cases} \tag{28}$$

where γ is the standard Gaussian probability on $\mathbb{R}$: $d\gamma(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\xi^2}{2}\right] d\xi$.

Moreover, the covariance of $(\eta(s), \eta(t))$ under μ_T is $c(s,t)$ if $s \neq t$. Thus in this case, considering the standard integration formula of a 2 dimensional Gaussian vector:

$$E[f(X)g(Y)] =$$
$$\int \int f\left(\sqrt{\frac{Var(X)Var(Y)-Cov(X,Y)^2}{Var(Y)}}\xi_1 + \frac{Cov(X,Y)}{\sqrt{Var(Y)}}\xi_2 + E(X)\right) g[\sqrt{Var(Y)}\xi_2 + E(Y)]d\gamma(\xi_1)d\gamma(\xi_2)$$

we obtain the following evolution equation for covariance:

$$c(s+1, t+1) =$$
$$J^2 \int \int f\left(\sqrt{\frac{[q(s)+\sigma^2][q(t)+\sigma^2]-c(s,t)^2}{q(t)+\sigma^2}}\xi_1 + \frac{c(s,t)}{\sqrt{q(t)+\sigma^2}}\xi_2 + m(s) - \theta\right) f[\sqrt{q(t)+\sigma^2}\xi_2 + m(t) - \theta]d\gamma(\xi_1)d\gamma(\xi_2)$$
$$\tag{29}$$

The dynamics of the mean-field system (28,29) can be studied as a function of the parameters:

- the mean $\bar{J}$ of the connexion weights,
- the standard deviation J^2 of the connexion weights
- the firing threshold θ of neurons.

Notice that the time and size limits do not necessarily commute. Therefore, any result on long time dynamics of the mean-field system may not be an exact prediction of the large-size limit of stationary dynamics of random recurrent networks. However, for our model, extensive numerical simulations have shown ([12],[17] and chapter I) that the time asymptotics of the mean-field system is informative about moderately large random recurrent network stationary dynamics (from size of some hundred neurons).

More precisely, in the low noise limit ($\sigma << 1$), two points of view are interesting:

- the ensemble stationary dynamics is given by the study of the time asymptotics of the dynamical system

$$\begin{cases} m(t+1) = \bar{J} \int f[\sqrt{q(t)}\xi + m(t) - \theta]d\gamma(\xi) \\ q(t+1) = J^2 \int f[\sqrt{q(t)}\xi + m(t) - \theta]^2 d\gamma(\xi) \end{cases} \tag{30}$$

- the synchronization of the individual neuron trajectories. Actually, $m(t)$ and $q(t)$ may converge, when $t \to \infty$, towards limits m^* and q^* (stable equilibria of the dynamical system 30) with a great variety of dynamical behaviors. Each individual trajectory may converge to a fixed point and (m^*, q^*) are the statistical moments of the fixed point empirical distributions. Another case is provided by individual chaotic oscillations around m^* where q^* measures the amplitude of the oscillations.

The discrimination between these two situations which are very different from the point of view of neuron dynamics is given by the study of the mean quadratic distance which will be outlined in the next paragraph.

4.1.2 Study of the mean quadratic distance

The concept of mean quadratic distance was introduced by Derrida and Pommeau in [19] to study the chaotic dynamics of extremely diluted large size networks. The method originates to check the sensitivity of the dynamical system to initial conditions. The idea is the following: let us consider two networks trajectories $u^{(1)}$ and $u^{(2)}$ of the same network configuration which is given by the synaptic weight matrix (J_{ij}). Their mean quadratic distance is defined by

$$d_{1,2}(t) = \frac{1}{N} \sum_{i=1}^{N} [u_i^{(1)}(t) - u_i^{(2)}(t)]^2$$

For a given configuration, if the network trajectory converges towards a stable equilibrium or towards a limit cycle (synchronous individual trajectories), then the mean quadratic distance between closely initialized trajectories goes to 0 when times goes to infinity. On the contrary, when this distance goes far from 0, for instance converges towards a non zero limit, whatever close the initial conditions are, the network dynamics present in some sense "sensitivity to initial conditions" and thus this behavior of the mean quadratic distance can be considered to be symptomatic of chaos. We applied this idea in [11] to characterize instability of random recurrent neural network.

In the context of large deviation based mean-field theory, the trajectories $u^{(1)}$ and $u^{(2)}$ are submitted to independent synaptic noises and the mean quadratic distance is defined by

$$d_{1,2}(t) = \frac{1}{N} \sum_{i=1}^{N} \int [u_i^{(1)}(t) - u_i^{(2)}(t)]^2 dQ_N^{(1,2)}(u^{(1)}, u^{(2)}) \tag{31}$$

where $Q_N^{(1,2)}$ is the joint probability law on $\mathcal{F}^{2N}$ of the network trajectories $(u^{(1)}, u^{(2)})$ over the time interval $\{0, ..., T\}$. Following the same lines as in last sections, it is easy to show a large deviation principle for the empirical measure of the sample $(u_i^{(1)}, u_i^{(2)})_{i \in \{1,...,N\}}$ under $Q_N^{(1,2)}$ when $N \to \infty$. Then we get the almost sure convergence theorem

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \int f_1(u_i^1) f_2(u_i^2) dQ_N(\mathcal{J})(u) = \int f_1(\eta_1) f_2(\eta_2) d\mu_T^{(1,2)}(\eta_1, \eta_2)$$

where $\mu_T^{(1,2)}$ is the fixed point of the mean-field evolution operator $L^{(1,2)}$ of the joint trajectories, which is defined on the probability measure set $\mathcal{P}(\mathcal{F} \times \mathcal{F})$ exactly in the same way as L was defined previously in definition 3.

Then if we define the instantaneous covariance between two trajectories by:

Definition 5 *The instantaneous cross covariance between the two trajectories under their joint probability law is defined by*

$$c_{1,2}(t) = \int \eta_1(t) \eta_2(t) d\mu_T^{(1,2)}(\eta_1, \eta_2) \tag{32}$$

where $\mu_T^{(1,2)}$ is the fixed point measure of the joint evolution operator $L^{(1,2)}$ defined from an initial condition $\mu_{init}^{(1,2)}$,

then we can follow the argument, which was already used for the covariance evolution equation (29). Thus we obtain the following evolution equation for the instantaneous cross covariance equation

$$c_{1,2}(t+1) =$$
$$J^2 \int \int f \left(\sqrt{\frac{[q_1(t)+\sigma^2][q_2(t)+\sigma^2]-c_{1,2}(t)^2}{q_2(t)+\sigma^2}} \xi_1 + \frac{c_{1,2}(t)}{\sqrt{q_2(t)}} \xi_2 + m_1(t) - \theta \right) f[\sqrt{q(t)+\sigma^2} \xi_2 + m_2(t) - \theta] d\gamma(\xi_1) d\gamma(\xi_2)$$
$$\tag{33}$$

The proof is detailed in [33].

It is obvious now to infer the evolution of the mean quadratic distance from the following square expansion.

Proposition 13 *The mean quadratic distance obeys the relation*

$$d_{1,2}(t) = q_1(t) + q_2(t) - 2c_{1,2}(t) + [m_1(t) - m_2(t)]^2 \tag{34}$$

4.1.3 Study of the special case of balanced inhibition

In order to show how the previous equations are used we shall display the special case of balanced inhibition and excitation. The study of the discrete time 1-dimensional dynamical system with different parameters was addressed in paper I. See also ([12] and [17]) for more details.

We choose in the previous model the special case where $\bar{J} = 0$. This choice simplifies considerably the evolution study since $\forall t, m(t) = 0$ and the recurrence over $q(t)$ is autonomous. So we have just to study the attractors of a single real function.

Moreover, the interpretation of $\bar{J} = 0$ is that there is a general balance in the network between inhibitory and excitatory connections. Of course, the model is still far from biological plausibility since the generic neuron is endowed both with excitatory and inhibitory functions. In next section, the model with several populations will be addressed. Nevertheless, the case $\bar{J} = 0$ is of special interest. In the limit of low noise, the mean-field dynamical system amounts to the recurrence equation:

$$q(t+1) = J^2 \int f[\sqrt{q(t)}\xi - \theta]^2 d\gamma(\xi) \tag{35}$$

we can scale $q(t)$ to J^2 and we obtain

$$q(t+1) = \int f[J^2\sqrt{q(t)}\xi - \theta]^2 d\gamma(\xi) = h_{J^2,\theta}[q(t)] \tag{36}$$

where the function $h_{J^2,\theta}$ of $\mathbb{R}^+$ into $\mathbb{R}^+$ is defined by

$$h_{J^2,\theta}(q) = \int f[J^2\sqrt{q(t)}\xi - \theta]^2 d\gamma(\xi)$$

This function is positive, increasing and tends to 0.5 when q tends to infinity. The recurrence (36) admits on $\mathbb{R}^+$ a single stable fixed point $q^*(J^2, \theta)$. This fixed point is increasing with J^2 and decreasing with θ. We represent in figure 1 the diagram of the variations of function $q^*(J^2, \theta)$. It is obtained from a numerical simulation with a computation of $h_{J^2,\theta}$ by Monte-Carlo method.

Let us now consider the stability of the network dynamics by studying the covariance and the mean quadratic distance evolution equation. The covariance evolution equation (29) in the low noise limit and when $t \to \infty$ amounts to

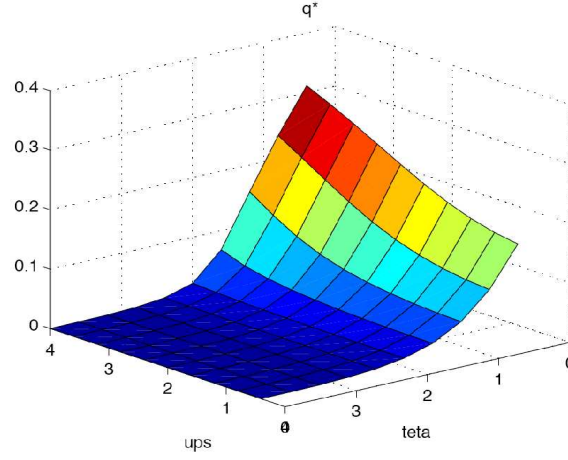$$c(s+1, t+1) = J^2 \int\int f\left(\sqrt{\frac{q^{*2}-c(s,t)^2}{q^*}}\xi_1 + \frac{c(s,t)}{\sqrt{q^*}}\xi_2 - \theta\right) f\left(\sqrt{q^*}\xi_2 - \theta\right) d\gamma(\xi_1)d\gamma(\xi_2) \tag{37}$$
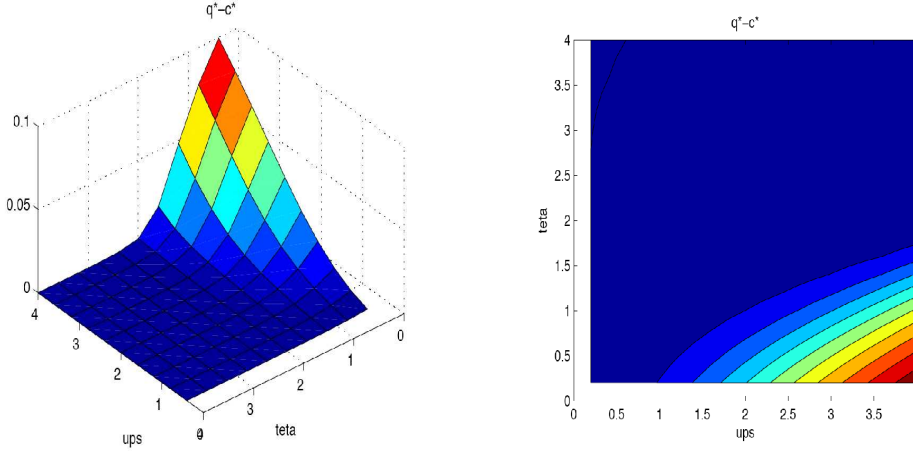
Scaling the covariance with J^2 we obtain the recurrence

$$c(s+1, t+1) = H_{J^2,\theta,q}[c(s,t)]$$

with

$$H_{J^2,\theta,q}(c) = \int\int f\left(J^2\sqrt{\frac{q^2-c^2}{q}}\xi_1 + \frac{c}{\sqrt{q}}\xi_2 - \theta\right) f\left(J^2\sqrt{q}\xi_2 - \theta\right) d\gamma(\xi_1)d\gamma(\xi_2) \tag{38}$$

Fig. 1. Variations of the fixed point $q^*(J^2, \theta)$ as a function of the network configuration parameters

Fig. 2. Variations of $q^* - c^*$ as a function of the network configuration parameters J^2 and θ

It is clear from comparing with equation (35) that q^* is a fixed point of $H_{J^2,\theta,q}$. To study the stability of this fixed point, standard computation shows that

$$\frac{dH_{J^2,\theta,q^*}}{dc}(q^*) = \int f' \left(J^2 \sqrt{q^*} \xi_2 - \theta\right)^2 d\gamma(\xi) \tag{39}$$

Then, as it is stated in paper I, the condition $\frac{dH_{J^2,\theta,q^*}}{dc}(q^*) \leq 1$ is a necessary and sufficient condition for the stability of q^*. A detailed and rigorous proof for $\theta = 0$ is provided in [33]. Then two cases occur.

– In the first case where $\frac{dH_{J^2,\theta,q^*}}{dc}(q^*) \leq 1$, the stationary limit of $c(s + \tau, t + \tau)$ when $\tau \to \infty$ does not depend on $t - s$ and is $c^* = q^*$. The stationary limit of the mean-field Gaussian process is a random point. Its variance is increasing with J^2 and decreasing with θ.
– In the second case where $\frac{dH_{J^2,\theta,q^*}}{dc}(q^*) > 1$ does not depend on $t - s$ when $t - s \neq 0$ and is equal to $c^* < q^*$. The stationary limit of the Gaussian process is the sum of a random point and of a white noise. From the dynamical system point of view, this corresponds to

a chaotic regime (with infinitely many degrees of freedom). The signature of chaos is given by the evolution of the mean quadratic distance. The instantaneous covariance converges also towards c^*. Therefore the mean quadratic distance converges towards a non zero limit, which is independent of the initial condition distance. As shown in [11] the transition from fixed point to chaos is given by an explicit equation which is the same as the equation of the De Almeida-Thouless line [5] in spin-glasses models. The analogy between these two systems is further developed in [6,11].

The figures 1 and 2 shows the evolution of q^* and $q^* - c^*$ as a function of J^2 and θ. When J^2 is small, there is no bifurcation. When J^2 is larger, a transition to chaos occurs when θ is decreasing. When J^2 is growing, the transition to chaos occurs for increasing θ values. Figure 31 of paper I shows the interest of variation of input (which is equivalent to threshold variation) allows to hold up the occurence of chaos.

4.2 Mean-field dynamics of 2-population AFRRNN

4.2.1 2-population AFRRNN model

As it was announced previously, the assumption of a homogeneous connexion weight model is not plausible. Besides, in literature, RRNN models with several neuron populations have been studied as early as in 1977 with [2] and have been thoroughly investigated in the last decade (see for instance [27]). The heterogeneity of neuron population induces interesting and complex dynamical phenomena such as synchronization. Actually the mean-field theory that was developed in the previous sections may be extended without major difficulty to several neuron populations. To give a practical idea of what can be obtained such extensions we consider here two populations with respectively $N_1 = \lambda N$ and $N_2 = (1 - \lambda)N$ neurons where $\lambda \in]0, 1[$ and where $N \to \infty$.

Four connexion random matrices have to be considered in this model $\mathcal{J}_{11}, \mathcal{J}_{12}, \mathcal{J}_{21}, \mathcal{J}_{22}$ where $\mathcal{J}_{ij}$ is the matrix of connexion weights from population j to population i. The random matrix $\mathcal{J}_{ij}$ is a $(N_j \times N_i)$ random matrix with independent identically distributed entries. Their distribution is governed by statistical parameters $(\bar{J}_{ij}, J_{ij}^2)$ and obeys hypothesis (9). They are independent altogether.

However, the technical hypothesis (H) does not allow us to give to connexion weights a rigorously constant sign, permitting to distinguish between inhibitory and excitatory neurons. Indeed, there is no probability distribution on *positive* (resp. negative) real numbers, having a mean and a variance respectively scaling as $\frac{\bar{J}}{N}$ and $\frac{J^2}{N}$. Thus, the positivity of the support induces on the other side of the distribution a heavy tail which will not respect assumption (iii) in hypothesis (H). However, it is possible to consider probability distributions which are checking hypothesis (H) and which are loading the negative numbers (or alternatively) the positive ones) with arbitrary small probability.

We consider here a 2-population model with a population of excitatory neurons and a population of inhibitory neurons (up to the above restriction).

4.2.2 General mean-field equations for moments

A large deviation principle may be obtained for the 2-population model for Gaussian connexion weights. So, the convergence in finite time to the mean-field dynamics is shown, in the present model with the same proof as in the previous 1-population model. See [33] for a rigorous proof and [16] for a more practical statement of results. The limit of the empirical measure is the law of a Gaussian vector which takes its values in $\mathcal{F} \times \mathcal{F}$. Each factor stands to describe the repartition of a neural population. Note that the two components are independent. As for the 1-population model we note $m_k(t), q_k(t), c_k((s, t)$ the mean, variance and covariance at given times of the empirical measure of population k ($k \in \{1, 2\}$). The mean-field evolution equation for these moments is described by the following system:

$$
\begin{cases}
m_k(t+1) = \sum_{j \in \{1,2\}} \bar{J}_{kj} \int f[\sqrt{q_j(t)+\sigma^2}\xi + m_j(t) - \theta_j]d\gamma(\xi) \\
q_k(t+1) = \sum_{j \in \{1,2\}} J_{kj}^2 \int f[\sqrt{q_j(t)+\sigma^2}\xi + m_j(t) - \theta_j]^2 d\gamma(\xi) \\
c_k(s+1,t+1) = \sum_{j \in \{1,2\}} J_{kj}^2 \int f\left(\sqrt{\frac{[q_k(s)+\sigma^2][q_k(t)+\sigma^2]-c_k(s,t)^2}{q_k(t)+\sigma^2}}\xi_1 + \frac{c_k(s,t)}{\sqrt{q_k(t)+\sigma^2}}\xi_2 + m_k(s) - \theta_k \right) \times \\
\times f[\sqrt{q_k(t)+\sigma^2}\xi_2 + m_k(t) - \theta_k]d\gamma(\xi_1)d\gamma(\xi_2)
\end{cases}
$$

$$(40)$$

4.2.3 Results and discussion

As far as numerical studies are concerned, we choose the following values for the statistical parameters

$$
\begin{cases}
\bar{J}_{1,1} = \ gd & J_{1,1} = \ g \\
\bar{J}_{1,2} = -2gd & J_{1,2} = \sqrt{2}g \\
\bar{J}_{2,1} = \ gd & J_{2,1} = \ g \\
\bar{J}_{22} = \ 0 & J_{22} = \ 0
\end{cases}
$$

$$(41)$$

In this study, according to some biological scheme, excitatory neurons are connected both to excitatory neurons and inhibitory neurons and inhibitory neurons are both connected to excitatory neurons. Moreover, the number of parameters is reduced to allow numerical exploration of the synchronization parameter. We keep two independent parameters:

− g stands for the non linearity of the transfer function
− d stands for the differentiation of the two populations (inhibitory vs. excitatory).

Considering the firing thresholds as previously, there is no variation about individual thresholds. Excitatory neuron threshold θ_1 is chosen equal to 0 and inhibitory neuron threshold θ_2 is chosen equal to 0.3 because the activation potential of inhibitory neurons is always positive.

In the bifurcation map of 3 (extracted from [16]) several dynamical regimes are displayed and the corresponding numerical ranges of parameters d and g are displayed. Notice that theoretical previsions of the mean-field equations (40) and the large scale simulations of large-sized networks behavior are consistent.

As in the homogeneous case there is a transition to chaos for weak d. When the differentiation parameter d is sufficiently large (about 2), the fixed point looses its stability through a Hopf bifurcation to give rise to synchronous oscillations when g is growing. There is then a succession of bifurcations leading to chaos (see paper I).

Moreover, a new phenomenon occurs. For large g, there is a significant transition regime between stationary chaos and synchronized oscillations which is named "*cyclo-stationary chaos*". In that regime, statistical parameters are exhibiting regular periodic oscillations, though individual trajectories are diverging with a mean quadratic distance behaviour which is characteristic from chaos.

5 MFT-based oscillation analysis in IF networks.

In this section we would like to give an interesting application of mean-field approaches for spiking neurons. It was developed in [9]. This paper is part of a current of research which studies the occurrence of synchronized oscillations in recurrent spiking neural networks [4, 3, 8], in order to give an account of spatio-temporal synchronization effects which are observed in many situations in neural systems [24, 36, 10, 35].
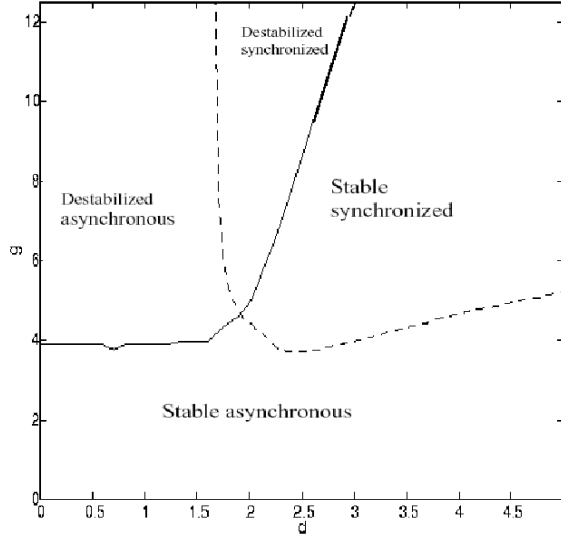
Fig. 3. Bifurcation map of the 2-population model

5.1 IFRRNN continuous-time model

The model of [9] has continuous time. There is no synaptic noise but the neurons are submitted to a random external output. So, equation (9) has to be replaced by

$$\begin{cases} u(t) < \theta & \Rightarrow \tau\dot{u}(t) = -u(t) + v_{net}(t) + v_{ext}(t) \\ u(t-0) = \theta \Rightarrow u(t+0) = \vartheta \end{cases} \tag{42}$$

where

- τ is the *characteristic time* of the neuron,
- v_{net} is the synaptic input from the network,
- v_{ext} is the external input,
- ϑ is the reset potential and $0 < \vartheta < \theta$. Note that $u(t-0)$ and $u(t+0)$ are respectively the left and right limits of u at firing time t. Thus, the refractory period is assumed to be zero.

This model of continuous time neuron dynamics is introduced in paper I, section 2.2.4.

Moreover, since the inputs are modeled by continuous-time stochastic processes, equation (42) is a stochastic differential equation of the type

$$\tau du(t) = -u(t)dt + dV_t \tag{43}$$

with $dV(t) = dV_{ext}(t) + dV_{net}(t)$

Now we shall explicit these stochastic processes, in order to obtain the Fokker-Planck equation of the network dynamics, in a mean-field approximation.

5.2 Modeling the external input

The network is a recurrent inhibitory network and we study its reaction to random excitatory synaptic inputs. We suppose that in the network each neuron receives excitations from C_{ext} external neurons connected via constant excitatory synapses J_{ext}. The corresponding external current is a Poisson process with emission frequency ν_{ext}.

Let us examine the effect of a superimposition of a large number C of independent identically distributed low-rate ν Poisson processes. Put

$$\mathcal{I}(t) = J \sum_{i=1}^{C} \mathcal{N}_i(t)$$

where $\mathcal{N}_i(t)$ are i.i.d. Poisson processes with firing rate ν. Then $I(t)$ is a stochastic process with independent stationary increments such that $\mathrm{E}(\mathcal{I}(t)) = \mu t = JC\nu t$ and $\mathrm{Var}(\mathcal{I}(t)) = \sigma^2 t = J^2 C \nu t$. Thus $\mu = JC\nu$ and $\sigma = J\sqrt{C\nu}$.

We are interested in studying such processes when they reach the firing threshold θ which is far larger than the elementary increment J. In typical neural applications, $J = 0.1$ mv and $\theta = 20$ mV. At this level, operating a classical time-space rescaling, $\mathcal{I}(t)$ appears like a Gaussian process with independent increments and same moments. We have

$$d\mathcal{I}(t) \sim \mu dt + \sigma dB_t$$

where (B_t) is the standard Brownian motion. If we apply the subsequent to the external synaptic input we get the following modeling in the limit of large size and low rate

$$dV_{ext}(t) = \mu_{ext} dt + \sigma_{ext} dB(t)$$

with $\mu_{ext} = J_{ext} C_{ext} \nu_{ext}$ and $\sigma_{ext} = J_{ext} \sqrt{C_{ext} \nu_{ext}}$.

5.3 Mean-field approximation of the internal input

In the framework of continuous-time modeling, the synaptic input definition of v_{net} for IF neuron i which was, according to equation (3),

$$v_i(\mathcal{J}, u)(t+1) = \sum_{j=1}^{N} J_{ij} x_j(t),$$

has to be replaced by

$$v_i(\mathcal{J}, u)(t) = \tau \sum_{j=1}^{N} J_{ij} \sum_{k} \delta\left(t - T_j^k(u) - D\right) \tag{44}$$

where

- δ is the Dirac distribution,
- $T_j^k(u)$ are the successive firing times of neuron j during the network trajectory u,
- D is the synaptic transmission delay.

In the present study, the network is supposed to be sparsely connected. All the connexion weights are equal to $-J$ as soon as they are non zero. Each neuron is connected to C neurons which are randomly drawn among the network with $C << N$ connections, where C is a fixed integer and N is the total number of neurons. Another model is considered further where the connection weights are independent random variables equal to $-J$ with probability $\frac{C}{N}$ and to 0 else. We shall focus here on the first model.

In previous sections, mean-field approximation in the finite time set framework consisted in finding a fixed point for the mean-field propagation operator L. Namely:

- approximating random vectors v_i by Gaussian vectors with a probability distribution g_μ, where μ is a probability law on the individual neuron potential trajectory space (finite-dimensional vector space)

– finding μ as the probability law of the neuron dynamical equation with this approximation for the synaptic input.

The mean-field approximation in [9] follows the same logic.

The first step of the mean field approximation consists for a given rate function ν in defining the non stationary Gaussian process

$$dV_{net}(t) = \mu_{net}(t)dt + \sigma_{net}(t)dB(t) \tag{45}$$

where

– the drift μ_{net} is given by

$$\mu_{net}(t) = -CJ\nu(t - D)\tau \tag{46}$$

– and where the diffusion coefficient σ_{net} is given by

$$\sigma_{net}(t)^2 = J^2 C\nu(t - D)\tau \tag{47}$$

The second step consists in considering the following diffusion with "tunneling effect"

$$\begin{cases} u(t) < \theta & \Rightarrow \tau du(t) = -u(t)dt + dV_{net}(t) + dV_{ext}(t) \\ u(t - 0) = \theta \Rightarrow u(t + 0) = \vartheta \end{cases} \tag{48}$$

The terminology "tunneling effect", referring to quantum mechanics, is somewhat curious here. It has its roots in the following remark. Whenever the membrane potential reaches θ it is reset to ϑ. If we interpret eq. (48) in the context of a random particle motion, the "particle" is "instantaneously transported" from the point θ to ϑ. This analogy is not only formal. The "tunneling effect" induces a specific behavior for the probability current at the boundary $u = \theta$. In the present model, this current is directly related to the firing rate (see next section).

5.4 Fokker-Planck equation.

5.4.1 Closed form equation

Note $p(u, t)$ the probability density of the solution $u(t)$ of (48). Define

$$\mu(t) = \mu_{net}(t) + \mu_{ext}(t)$$
$$\sigma(t) = \sqrt{\sigma_{net}(t)^2 + \sigma_{ext}(t)^2}$$

Then $p(u, t)$ is solution of the Fokker-Planck equation for diffusion process for $u < \theta$ and $u \neq \vartheta$:

$$\frac{\partial p}{\partial t}(u, t) = \frac{\sigma(t)^2}{2} \frac{\partial^2 p}{\partial u^2}(u, t) + \frac{\partial}{\partial u}[(u - \mu(t))p(u, t)] \tag{49}$$

The tunneling effect from θ to ϑ is taken into account in the following boundary conditions

$$\begin{cases} p(\theta, t) = 0 \\ \frac{\partial p}{\partial u}(\vartheta + 0, t) = \frac{\partial p}{\partial u}(\vartheta - 0, t) + \frac{\partial p}{\partial u}(\theta - 0, t) \end{cases} \tag{50}$$

This corresponds to a re-injection of the outgoing probability current $j(\theta, t)$ at $u = \vartheta$, where $j = \frac{\partial p}{\partial u}$. Thus $j(\vartheta + 0, t) = j(\vartheta - 0, t) + j(\theta, t)$. The outgoing current (re-injected current) is, by definition, the average firing rate defined by

$$\nu(t) = \frac{\partial p}{\partial u}(\theta - 0, t) \tag{51}$$

5.4.2 Stationary solution

It is easy to find the stationary solution of the previous equation

$$\frac{\partial p}{\partial t}(u, t) = 0$$

Suppose a given constant firing rate ν_0, then set

$$\begin{cases} \mu_0 = -CJ\nu_0\tau + \mu_{ext} \\ \sigma_0 = \sqrt{CJ^2\nu_0\tau + \sigma_{ext}^2} \end{cases} \tag{52}$$

and plug it into the differential second order equation

$$\frac{\sigma_0^2}{2}\frac{d^2p}{du^2} + \frac{d}{du}[(u - \mu_0)p(u)] = 0 \tag{53}$$

with the following boundary conditions

$$\begin{cases} p(\theta) = 0 \\ \frac{dp}{du}(\vartheta + 0) = \frac{dp}{du}(\vartheta - 0) + \frac{dp}{du}(\theta - 0, t) \end{cases} \tag{54}$$

One obtains easily the following stationary distribution

$$\begin{cases} \text{For } u < \vartheta, \ p(u) = \frac{2\nu_0}{\tau}e^{-y_u^2}\int_{J_\vartheta^2}^{J_\theta^2} e^{y^2}dy \\ \text{For } u \geq \vartheta, \ p(u) = \frac{2\nu_0}{\tau}e^{-y_u^2}\int_u^{v_\theta} e^{y^2}dy \end{cases}$$

where $y_u = \frac{u - \mu_0}{\sigma_0}, y_\vartheta = \frac{\vartheta - \mu_0}{\sigma_0}$ and $y_\theta = \frac{\theta - \mu_0}{\sigma_0}$

Then the normalization condition $\int_{-\infty}^{\infty} p(u)du = 1$ allows to infer

$$\frac{1}{\nu_0\tau} = \int_0^{+\infty} e^{-y^2}\left[\frac{e^{2y_\theta y} - e^{2y_\vartheta y}}{y}\right]dy \tag{55}$$

The relations (52,55) allows to compute numerically ν_0. The equation (55) can be approximately solved in the situation where the fluctuations σ_0 are weak (i.e. $y_\theta \gg 1$ which means that the spiking events are rare). In this case :

$$\nu_0\tau \approx \frac{y_\theta}{\sqrt{\pi}}e^{-y_\theta^2} \tag{56}$$

This asymptotic expression can be compared to the escape probability from the equation of motion of a particle in a parabolic potential well $\mathcal{V}$, with minimum μ_0, submitted to a Brownian excitation
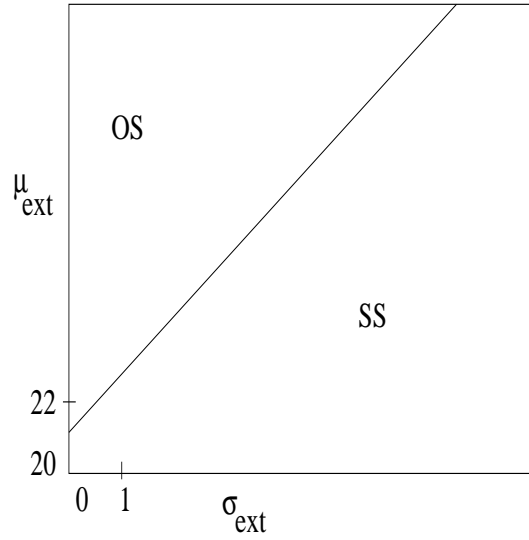
$$\tau dV_t = -(V - \mu_0)dt + \sigma_0 dB_t$$

The time rate to reach $V = \theta$ is thus given by the Arrhenius time

$$\nu_0\tau \sim e^{-y_\theta^2}$$

Numerical values of ν_0 which are inferred from equations (55) and (56) are compared in [9] to the result of numerical simulations of the network and there is a good agreement between theoretical predictions and simulated firing rates.

Fig. 4. Sketch of the bifurcation diagram of the model (10,44) when varying the parameters μ_{ext}, σ_{ext} controlling the Poisson process of external excitation. SS means Stationary State, while OS means Oscillatory State. The solid line represents the instability line for $D = 0.1\tau$. (Drawn by hand from [9])

5.4.3 Stability analysis.

The stability analysis for the stationary solution uses normal form techniques similar to those described in paper I, but in an infinite dimensional space. The Fokker-Planck equation is rescaled and expanded around the steady-state solution. This intricate computation is fully detailed in [9] . We simply focus to the results.

The Authors find that there is a bifurcation of Hopf type for the stationary solution. Thus, for a certain parameter range, the system exhibits synchronized oscillations of the neurons. A sketch of the bifurcation map is given in figure 4 when varying the parameters μ_{ext}, σ_{ext} controlling the external excitation.

One can see from that bifurcation diagram that the bifurcation occurs when the drift of the external input is increasing. On the opposite, an increase of the dispersion of the external input stabilizes the steady state. If the external input consists in the superposition of i.i.d. Poisson processes as it was detailed above, then the increase of their common frequency $\nu_e xt$ induces the occurrence of an oscillatory regime. There is still a good agreement between the predictions of mean-field theory and the results of simulations.

5.5 Conclusion

Thus, the conclusion is that in this model of a neural network with a sparsely connected inhibitory integrate-and-fire neurons, submitted to a external excitatory Poisson process, and emitting spikes irregularly at a low rate, there is, in the thermodynamic limit, a sharp transition between a regime where the average global is constant, to a synchronized state where neurons are weakly synchronized. The activity becomes oscillatory when the inhibitory feedback is strong enough. Note that the period of the global oscillations depends on the synaptic transmission delay which cannot be neglected.

Finally, let us mention that the Authors performed a finite size analysis of the model and found that global oscillations of *finite coherence time* generically exist *above and below* the critical inhibition threshold.

6 Appendix about probability theory

This paper uses intensively some classical notations and concepts coming from probability theory. The proofs are omitted but sometimes the results follow from advanced results of this theory. It is not possible to recall here the necessary prerequisites. There are excellent books about probability theory for physicists and engineers such as [30]. We just want here to recall some notations and some results from convergence theory. We have detailed the proof of the "finite-time Girsanov theorem" since it is a crucial result for the paper.

6.1 Elementary Notations

The classical and shortest point of view for considering random phenomena from the 19th century is to consider a random variable x in a space $\mathcal{E}$ via its probability law on that space. All the moments can be computed by integration over the probability law of the random variable. For instance, if μ is the probability law of the real random variable x, one has

$$\mathrm{E}(x) = \int_{\mathcal{E}} x d\mu(x)$$

$$\mathrm{E}(x^2) = \int_{\mathcal{E}} x^2 d\mu(x)$$

and more generally for any bounded continuous function ϕ of x

$$\mathrm{E}[\phi(x)] = \int_{\mathcal{E}} \phi(x) d\mu(x)$$

where E is the *mathematical expectation* operator. The expectation of any random variable is a vector in a topological vector space $\mathcal{F}$. The mathematical expectation operator is linear.

Moreover, for a random vector $x \in \mathbb{R}^d$ the expectation $E(x) \in \mathbb{R}^d$ is defined by

$$\forall i \in \{1, ..., n\}, \{\mathrm{E}(x)\}_i = \mathrm{E}(x_i) = \int_{\mathbb{R}^d} x_i d\mu(x)$$

and the symmetric $(d, d)-$covariance matrix is given by

$$\mathrm{Cov}(x)_{ij} = \mathrm{E}(x_i x_j) - \mathrm{E}(x_i)\mathrm{E}(x_j)$$

where μ is the probability law of x.

Actually, this point of view cannot be used when we are obliged to consider an infinite set of random variables or when we want to operate a variable change. Hence, we are obliged to adopt a more general point of view which was initiated by Kolmogorov in 1933. This approach relies basically upon the consideration of a very large state space Ω which describes all the possible outcomes or states of the world. Then a rich family $\mathcal{A}$ of subsets of Ω is defined such that all the random events of interest are belonging to $\mathcal{A}$. Eventually a probability measure is defined on $\mathcal{A}$ which associates to any random event $A \in \mathcal{A}$ its probability $P(A)$. The triple $(\Omega, \mathcal{A}, P)$ is called a *probability space*.

Later on, we shall have to work on infinite-dimensional space. So let us fix a general framework

Definition 6 *A Polish space $\mathcal{F}$ is a metric complete (every Cauchy sequence converges) and separable (there is a countable dense subset) space. The $\sigma-$algebra $\mathcal{B}$ of Borel subsets of A Polish space $\mathcal{F}$ is the smallest $\sigma-$algebra that contains the open sets. Given a probability measure μ on the Borel subsets of $\mathcal{F}$ it is possible to integrate any bounded continuous function ϕ on $\mathcal{F}$ and the integral is noted $\int_{\mathcal{F}} \phi(\xi) d\mu(\xi)$. The integral may be extended to a wider class of functions. These functions are called <u>integrable</u> with respect to μ.*

In that new framework let us define random variables in $\mathcal{F}$.

Definition 7 *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{F}, \mathcal{B})$ a Polish space endowed with its Borel $\sigma-$algebra. A random variable $x \in \mathcal{F}$ is a state function from Ω into $\mathcal{F}$ such that for any open set B in $\mathcal{F}$, the subset of Ω defined by*

$$(x \in B) = \{\omega \in \Omega \text{ such that } x(\omega) \in \mathcal{B}\}$$

belongs to $\mathcal{A}$ so its probability $P(x \in B)$ is well defined.

The probability law of a random variable $x \in \mathcal{F}$ is the probability law on $\mathcal{F}$ which associates to any Borel subset $B \subset \mathcal{F}$ the probability $P(x \in B)$.

The probability law of x is noted $x.P$ or P_x. This definition stands for also for general measure than probability laws such as volume measures. More generally, we have

Definition 8 *Let $(\Omega, \mathcal{A}, P)$ be a measure space and x a mapping from Ω to $\mathcal{F}$ such that*

$$\forall B \in \mathcal{B}, (x \in B) = \{\omega \in \Omega \text{ such that } = x(\omega) \in \mathcal{B}\} \in \mathcal{A}$$

Then we define a measure on $(\mathcal{F}, \mathcal{B})$ that is noted $x.P$ or P_x by

$$B \in \mathcal{B} \rightarrow x.P(B) = P_x(B) = P(x \in B)$$

This measure is called the __image__ of the measure P by the mapping x

This definition is completed by the following transfer theorem which shows that the mathematical expectation can be computed on the state space Ω or on the value space $\mathcal{F}$.

Theorem 14 *For any function ϕ defined on $\mathcal{F}$ and integrable for the probability law P_x we have*

$$E[\phi(x)] = \int_\Omega \phi[x(\omega)]dP(\omega) = \int_\mathcal{F} \phi(\xi)dP_x(\xi)$$

The transfer theorem is very useful in theory and in practice. It allows to define the mathematical expectation of a random variable without any ambiguity.

Kolmogorov's framework allows to define independent random variables by the equivalent following properties

Definition 9 *For $i \in \{1, ..., n\}$ let $x_i \in \mathcal{F}_i$ be random variables, they are said __independent__ if the law P_x of the random variable $x = (x_1, ..., x_n) \in \mathcal{F}_1 \times ... \times \mathcal{F}_n$ is the product of the P_{x_i} which is expressed in the following equivalent properties*

$$P(x \in B_1 \times ... \times B_n) = P_{x_1}(B_1)...P_{x_n}(B_n)$$

$$E[\phi_1(x_1)...\phi_n(x_n)] = E[\phi_1(x_1)]...E[\phi_n(x_n)]$$

6.2 Density and Gaussian random vectors

Definition 10 *Let $(\Omega, \mathcal{A}, m)$ a measure space and h an integrable positive function on Ω such that $\int_\Omega h(\omega)dm(\omega) = 1$. Then we can define a probability measure Q on $(\Omega, \mathcal{A})$ by*

$$Q(A) = \int_\Omega 1_A(\omega)h(\omega)dm(\omega)$$

Q is said __absolutely continuous__ with respect to m, h is called the __density__ of Q with respect to m and we can compute the integral for Q by using the formula

$$\int_\Omega \phi(\omega)dQ(\omega) = \int_\Omega \phi(\omega)h(\omega)dm(\omega)$$

We write $\frac{dQ}{dm}(\omega) = h(\omega)$ or $dQ((\omega) = h(\omega)dm(\omega)$

Of course, the density functions are commonly used in elementary probability. An important class of probability measures is the Gaussian probability family.

Definition 11 *Let $a \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$. The Gaussian probability measure $\gamma = \mathcal{N}(a, \sigma^2)$ is defined by its density with respect to the Lebesgue measure λ on $\mathbb{R}$, which is*

$$\frac{d\gamma}{d\lambda}(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\xi - m)^2}{2\sigma^2}\right]$$

Similarly, this definition can be extended to d-dimensional vector space and even to infinite-dimensional Hilbert space. Here, we just need the following

Definition 12 *Let $\overline{\theta} \in \mathbb{R}^d$ and K be a $d \times d$ symmetric positive matrix, then there exists one and one only probability measure on $\mathbb{R}^d$, which is called the <u>Gaussian probability</u> $\gamma = \mathcal{N}(\overline{\theta}, K)$ such that if γ is the probability law of the random vector $x \in \mathbb{R}^n$ then $\forall u \in \mathbb{R}^d$, the law of the random variable $u^t x$[12] is $\mathcal{N}(u^t\overline{\theta}, u^t K u)$.*

Proposition 15 *Let x be a random vector with regular Gaussian probability $\gamma = \mathcal{N}(\overline{\theta}, K)$ then we have*

$$\begin{cases} E(x) = \int \xi d\gamma(\xi) = \overline{\theta} \\ Cov(x) = E(xx^t) - E(x)E(x)^t = K \end{cases}$$

So a Gaussian law is completely determined by its expectation and its covariance matrix.

Definition 13 *With the previous notations, if K is invertible, γ is said to be <u>regular</u> and the density of γ with respect to the Lebesgue measure λ is*

$$\frac{d\gamma}{d\lambda}(\xi) = \frac{1}{\sqrt{(2\pi)^n Det(K)}} \exp\left[-\frac{(\xi - m)^t K^{-1}(\xi - m)}{2}\right] \tag{57}$$

A common property of the Gaussian family is its stability by linear transforms and translation. More precisely, we have

Proposition 16 *Let x a Gaussian random vector which takes its value in the vector space E and Λ a linear mapping of E into F. Then $y = \Lambda x$ is a Gaussian random vector in F and*

$$\begin{cases} E(y) = \Lambda E(x) \\ Cov(y) = \Lambda Cov(x)\Lambda^t \end{cases} \tag{58}$$

Proposition 17 *Let x a Gaussian random vector which takes its value in the vector space E and $a \in E$. Then $y = x + a$ is a Gaussian random vector in F and*

$$\begin{cases} E(y) = E(x) + a \\ Cov(y) = Cov(x) \end{cases} \tag{59}$$

Corollary 18 *Let x be a random vector with regular Gaussian probability $\gamma = \mathcal{N}(\overline{\theta}, K)$ and let $a \in \mathbb{R}^d$, then the law γ_a of $x + a$ is the regular Gaussian law $\mathcal{N}(\overline{\theta} + a, K)$ and its density with respect to γ can be written as follows*

$$\frac{d\gamma_a}{d\gamma}(\xi) = \exp\left[a^t K^{-1}(\xi - \overline{\theta}) - \frac{1}{2}a^t K^{-1}a\right] \tag{60}$$

PROOF : The formula is checked using an easy and straightforward computation from the expression of the Gaussian density ∎

It is interesting to note that it is possible to define Gaussian probability on an infinite-dimensional vector space though it is not possible to define Lebesgue measure. However, in that paper we just use finite-dimensional Gaussian probabilities. An interesting property of the Gaussian measure, which is crucial in this paper, is the following finite-dimensional version of the Girsanov theorem[23].

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[12] u^t is the transpose of column vector u, so $u^t x$ is the scalar product of vectors u and x

Theorem 19 *Let m_0 a probability measure on $\mathbb{R}^d$ and let $\mathcal{N}(\alpha, K)$ be a Gaussian regular probability on $\mathbb{R}^d$. Let T a positive integer and $\mathcal{E}_T = (\mathbb{R}^d)^{\{0,...,T\}}$ the space of finite time trajectories in $\mathbb{R}^d$. Let w a Gaussian random vector in $\mathcal{E}_T$ with law $m_0 \otimes \mathcal{N}(\alpha, K)^T$. Let ϕ and ψ two measurable applications of $\mathbb{R}^d$ into $\mathbb{R}^d$. Then we define the random vectors x and y in $\mathcal{E}$ by*

$$\begin{cases} x_0 = w_0 \\ x(t+1) = \phi[x(t)] + w(t+1) \end{cases}$$

$$\begin{cases} y_0 = w_0 \\ y(t+1) = \psi[y(t)] + w(t+1) \end{cases}$$

Let P and Q be the respective probability laws on $\mathcal{E}$ of x and y, then Q is absolutely continuous with respect to P and we have

$$\frac{dQ}{dP}(\eta) = \exp \sum_{t=0}^{T-1} \left\{ \begin{array}{l} -\frac{1}{2}\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\psi[(\eta(t)] - \phi[(\eta(t)]\} \\ +\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\eta(t+1) - \alpha - \phi[\eta(t)]\} \end{array} \right\} \quad (61)$$

PROOF : The proof is a recursion on T. It is easy to check (61) for $T = 1$. To reduce the expression let us write down

$$y_0^T = (y(0), ..., y(T)), \eta_0^T = (\eta(0), ..., \eta(T))$$

and

$$\Theta_T(\eta_0^T) = \sum_{t=0}^{T-1} \left\{ \begin{array}{l} -\frac{1}{2}\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\psi[(\eta(t)] - \phi[(\eta(t)]\} \\ +\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\eta(t+1) - \alpha - \phi[\eta(t)]\} \end{array} \right\}$$

Suppose (61) is true up to T and let us compute the density of y up to $T + 1$. Let h be a bounded continuous test function defined on $\mathcal{E}_{t+1}$. We have by conditioning with respect to y_0^T

$$\mathrm{E}\left[h(y(T+1), y_0^T)\right] = \int \mathrm{E}\left\{h(w(T+1) + \psi[\eta(T)], \eta_0^T)\right\} dQ(\eta_0^T)$$

where the expectation is taken with respect to $w(T+1)$, which is independent from y_0^T. Let us explicit the Gaussian law $\mathcal{N}(\alpha, K)$ and use the recursion hypothesis:

$$\mathrm{E}\left[h(y(T+1), y_0^T)\right] =$$
$$C_K \int \int h(\omega + \psi[\eta(T)], \eta_0^T) \exp\left\{-\frac{1}{2}(\omega - \alpha)^t K^{-1}(\omega - \alpha)\right\} \exp \Theta_T(\eta_0^T) d\omega dP(\eta_0^T)$$

where C_K is the classic normalization constant for the Gaussian law. Then let us perform the translation $\varpi = \omega + \psi[\eta(T)]$, it gives

$$\mathrm{E}\left[h(y(T+1), y_0^T)\right] =$$
$$C_K \int \int h(\varpi, \eta_0^T) \exp\left\{-\frac{1}{2}(\varpi - \alpha - \psi[\eta(T)])^t K^{-1}(\varpi - \alpha - \psi[\eta(T)])\right\} \exp \Theta_T(\eta_0^T) d\varpi dP(\eta_0^T)$$

To simplify notations let us write down $\zeta_T = \psi[\eta(T)] - \phi[\eta(T)]$, we have

$$\mathrm{E}\left[h(y(T+1), y_0^T)\right] =$$
$$C_K \int \int h(\varpi, \eta_0^T) \exp\left\{-\frac{1}{2}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)^t K^{-1}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)\right\} \exp \Theta_T(\eta_0^T) d\varpi dP(\eta_0^T)$$

Let us develop the quadratic form in the exponential

$$-\frac{1}{2}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)^t K^{-1}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)$$
$$= -\frac{1}{2}(\varpi - \alpha - \phi[\eta(T)])^t K^{-1}(\varpi - \alpha - \phi[\eta(T)]) - \frac{1}{2}\zeta_T^t K^{-1}\zeta_T + \zeta_T^t K^{-1}(\varpi - \alpha - \phi[\eta(T)])$$

So we have

$$\exp\left\{-\tfrac{1}{2}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)^t K^{-1}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)\right\}$$
$$= \exp\left\{\tfrac{1}{2}(\varpi - \alpha - \phi[\eta(T)])^t K^{-1}(\varpi - \alpha - \phi[\eta(T)])\right\} \exp\left\{-\tfrac{1}{2}\zeta_T^t K^{-1}\zeta_T + \zeta_T^t K^{-1}(\varpi - \alpha - \phi[\eta(T)])\right\}$$

We obtain a product of two exponentials. The first one combines itself with $C_K d\varpi dP(\eta_O^T)$ to give $dP(\eta_O^{T+1})$; the second one combines itself with $\exp\Theta_T(\eta_0^T)$ to give $\exp\Theta_{T+1}(\eta_0^{T+1})$. So we get eventually

$$\mathrm{E}\left[h(y_0^{T+1})\right] = \int h(\eta_0^{T+1}) \exp\Theta_{T+1}(\eta_0^{T+1}) dP(\eta_0^{T+1})$$

∎

6.3 Convergence of random variables

The definition of probability is based upon the law of large numbers (LLN). This last result may be roughly formulated as follows:

When (x_n) is an independent [13] sequence of random variables with the same probability law p with two first moments $c = \int x dp(x)$ and $k = \int x^2 dp(x)$ then the sequence of empirical averages $\overline{x}_n = \frac{\sum_{k=1}^n x_k}{n}$ converges towards c.

This statement is not precise. The convergence may have several senses. Some useful convergence concepts in probability theory are the convergence in law, the convergence in probability and the almost sure convergence. Let us recall their definition.

Definition 14 *Let (x_n) and x be random variables on a probability=20 space $(\Omega, \mathcal{A}, P)$. The sequence of random variables (x_n) is said to*

– *converge in law to x if and only if for any continuous bounded function h, $E[h(x_N)] \rightarrow E[h(x)]$ [14]*
– *converge in probability to x if and only if*

$$\forall \epsilon > 0, P(\mid x_n - x \mid \geq \epsilon) \rightarrow 0$$

– *converge almost surely to x if and only if*

$$\exists N \subset \Omega \text{ with } P(N) = 0 \text{ such that } \forall \omega \notin N, x_n(\omega) \rightarrow x(\omega)$$

These definitions are stronger and stronger. Almost sure convergence implies convergence in probability which implies in turn convergence in law. Most mean-field computations of mean-field equations in random neural networks use the convergence of Fourier transforms through a Laplace limit integral ensuring convergence in law.

However, from the point of view of practitioners, almost sure convergence is more pleasant because a single realization of the sequence (X_n) allows to check the convergence. To check the weaker convergence statements, a lot of realizations of the sequence are necessary.

Let us return to the law of large numbers. The convergence in probability of the sequence $(\overline{x}_n)$ is specially easy to show since $\mathrm{E}(\overline{x}_n) = c$ and $\mathrm{Var}(\overline{x}_n) = \frac{k-c^2}{n}$. Then one has just to write the Bienaymé-Tchebychev inequality

$$P(\mid \overline{x}_n - c \mid \geq \epsilon) \leq \frac{k - c^2}{n\epsilon^2}$$

But this convergence is not strong enough to show the almost sure convergence (the so-called strong large number law).

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[13] Such a sequence is called an i.i.d. sequence
[14] An equivalent condition is the convergence of their characteristic functions (or Fourier transforms): $\forall t \in \mathbf{R}, \mathrm{E}(\exp(itx_n)) \rightarrow \mathrm{E}(\exp(itx))$

6.4 Large deviation principle

6.4.1 Cramer's theorem

One way to obtain the strong law is to show that the convergence in probability occurs much faster than it appears from Bienaymé-Tchebychev inequality.

Actually the following theorem was obtained by Cramer in the late 30's:

Theorem 20 *Let (x_n) sequence of i.i.d. random variables with probability law μ such that $\int \xi d\mu(\xi) = \overline{\theta}$. Then we have*

$$\forall a > \xi, \ \frac{1}{n} \log P(\overline{x}_n > a) \to -I(a)$$

where

$$I(a) = \max_{p \in \mathbb{R}} pa - E[\exp(px)]$$

This theorem can be extended to more general settings. It is the subject of *large deviation* theory. Let us first consider the case of finite-dimensional random vectors [29]

The following proposition is easy to prove:

Proposition 21 *Let μ a probability law on $\mathbb{R}^d$ such that for all $p \in \mathbb{R}^d$, $\Lambda(p) = \log E[\exp(p^t x)]$ exists. The function Λ is called the log-generating function of μ. We define its Legendre transform Λ^* on $\mathbb{R}^d$ as follows:*

$$\Lambda^*(a) = \sup_{p \in \mathbb{R}^d} [(p^t a) - \Lambda(p)]$$

then

a) Λ^* is a convex function (with ∞ as a possible value)
b) $\forall a \in \mathbb{R}^d, \Lambda^*(a) \geq 0$
c) $a = \int \xi d\mu(\xi) \Leftrightarrow \Lambda^*(a) = 0$

  PROOF : a) is straightforward, since the supremum of convex functions is convex
  b) comes from Jensen's inequality.
  c) comes from $\Lambda(0) = 1$

∎

Then we can state the Cramer's theorem for i.i.d. sequence of finite-dimensional random vectors:

Theorem 22 *Cramer's theorem:*
  Let (x_n) be a sequence of i.i.d. random vectors with a probability distribution μ according to the assumption and the notations of the previous proposition. Then for any Borel subset B of $\mathbb{R}^d$, we have

$$- \inf_{a \in B^o} \Lambda^*(a) \leq \frac{1}{n}\underline{\lim}_{n \to \infty} \log[\mathbb{P}(\overline{x}_n \in B^o)] \leq \frac{1}{n}\overline{\lim}_{n \to \infty} \log[\mathbb{P}(\overline{x}_n \in \overline{B})] \leq - \inf_{a \in \overline{B}} \Lambda^*(a) \quad (62)$$

where B^o is the interior set of B (the greatest open subset of B) and $\overline{B}$ is the closure of B (the smallest closed extension of B).

A consequence of Cramer's theorem is that for any closed subset F in $\mathbb{R}^d$ such that $\inf_{a \in \overline{B}} \Lambda^*(a) > 0$, $\mathbb{P}(\overline{X}_n \in F)$ goes to 0 exponentially fast when $n \to \infty$ and that the rate of convergence depends only on the value of Λ^* at the point of F where Λ^* reaches its minimum. This point is called the dominating point. For regular probability distributions where Λ^* is strictly convex, defined and continuous around $\overline{\theta} = E(x)$, the exponential decay of finite deviations from the expectation (large deviations) and the strong law of large numbers are easy consequences.

6.4.2 Large deviation principle in an abstract setting

The convergence with an exponential rate is a general situation, which is characterized in the following general definitions:

Definition 15 *Let $\mathcal{E}$ be a Polish space and I be a lower semi-continuous function of $\mathcal{E}$ into $[0, \infty]$. I is called a rate function. If I possesses the property of compact level set, i.e.*

$$\forall \epsilon > 0, \{x \in \mathcal{E} \text{ such that } I(x) \leq \epsilon\} \text{ is compact}$$

then I is called a good rate function.

Definition 16 *Given a rate function I on a Polish space $\mathcal{F}$ and a sequence of probability measures Q_n on $\mathcal{F}$, if for any Borel subset B of $\mathcal{F}$,*

- *(Q_n) satisfies the large deviation minoration on open sets if*

$$\forall O, \text{ open set in } \mathcal{F}, -\inf_{\xi \in O} I(\xi) \leq \frac{1}{n} \underline{\lim}_{n \to \infty} \log[Q_n(O)] \tag{63}$$

- *(Q_n) satisfies the $\underline{\text{large deviation majoration on compact sets}}$ if*

$$\forall K, \text{ compact set in } \mathcal{F}, \frac{1}{n} \overline{\lim}_{n \to \infty} \log[Q_n(K)] \leq -\inf_{\xi \in K} I(x) \tag{64}$$

- *(Q_n) satisfies the $\underline{\text{large deviation majoration on closed sets}}$ if*

$$\forall C, \text{ closed set in } \mathcal{F}, \frac{1}{n} \overline{\lim}_{n \to \infty} \log[Q_n(C)] \leq -\inf_{\xi \in C} I(x) \tag{65}$$

- *If (Q_n) checks the large deviation minoration for open sets and the large deviation majoration for compact sets we say that (Q_n) satisfies the $\underline{\text{large deviation principle (LDP)}}$ with rate function I.*
- *If (Q_n) checks the large deviation minoration for open sets and the large deviation majoration for closed sets we say that (Q_n) satisfies the $\underline{\text{full large deviation principle}}$ with rate function I.*
- *(Q_n) is said $\underline{\text{tight}}$ if for all $\epsilon > 0$, it exists a compact subset K of $\mathcal{F}$ such that $Q_n({}^{c}K) < \epsilon$. If (Q_n) is tight and checks a LDP, it satisfies the full LDP for the same rate function.*

The same definitions stand for a sequence of random elements in $\mathcal{F}$ if the sequence of their probability laws checks the respective majorations.

A simpler way to state that (Q_n) satisfy the full large deviation principle with rate function I is to write that

$$-\inf_{\xi \in B^o} I(\xi) \leq \frac{1}{n} \underline{\lim}_{n \to \infty} \log[Q_n(B)] \leq \frac{1}{n} \overline{\lim}_{n \to \infty} \log[Q_n(B)] \leq -\inf_{\xi \in \overline{B}} I(x) \tag{66}$$

Actually, the scope of Cramer's theorem may be widely extended and a full large deviation principle is checked for the empirical mean of any i.i.d. random sequence in a Polish space under mild assumptions on the existence of the log-generating function [18]. The rate function of this LDP is the Legendre transform of the log-generating function.

6.4.3 Varadhan theorem and Laplace principle

An equivalent functional formulation of the full large deviation principle is due to Varadhan and is called by Dupuis and Ellis the *Laplace principle* ([20]).

Definition 17 *Let I be a good rate function on the Polish space $\mathcal{F}$. The random sequence (x_n) in $\mathcal{F}$ is said to satisfy the* Laplace principle *with rate function I if for any continuous bounded function h on $\mathcal{E}$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \log E\{\exp[-nh(x_n)]\} = -\inf_{\xi\in\mathcal{F}}\{h(\xi) + I(\xi)\}$$

This approach is called by the authors of ([20]) the *weak convergence approach* to the theory of large deviations. The equivalence of the two approaches (Laplace principle for good rate functions and full large deviation principle with good rate functions) are expressed in a theorem of Varadhan and its converse. Their proofs are in ([20]). Handling continuous bounded test functions may be more practical than dealing with open and closed sets. In particular, it is very easy to show the following transfer theorem for the LDP principle when the law is changed.

Theorem 23 *Let P_n and Q_n two sequences of probability measures on the Polish space $\mathcal{F}$, let I be a good rate function on F and let Γ a continuous function on F such that*

(a) $Q_n << P_n$ and $\frac{dQ_n}{dP_n}(\xi) = \exp n\Gamma(\xi)$,
(b) (P_n) satisfies a full large deviation principle with rate function I,
(c) $I - \Gamma$ is a good rate function,

then (Q_n) satisfies a full large deviation principle with rate function $I - \Gamma$.

PROOF OF THEOREM: Using the weak large deviation approach and the strong hypothesis of the theorem, the proof is quite formal. Let h be any continuous bounded test function on $\mathcal{F}$, from hypothesis (c) and ∎

6.5 Convergence of random measures

Let us have a second look at the law of large numbers. Since this law claims the convergence on the sequence of empirical averages $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ over any bounded continuous test function f we are lead to consider the empirical measure of a sample.

Definition 18 *Let $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^{nd}$ a sequence of n vectors of=20 $\mathbb{R}^d$. We associate to ξ the following probability measure $\mu_\xi \in \mathcal{P}(\mathbb{R}^d)$*

$$\mu_\xi = \frac{1}{n}\sum_{k=1}^{n} \delta_{\xi_k}$$

$\mu_x i$ *is called the* empirical measure *associated to ξ.*

This definition says that if A is a Borel subset of $\mathcal{F}$ then $\mu_N(x)(A)$ is the fraction of neurons which state trajectory belong to A. More practically, if ϕ is any test continuous function on $\mathcal{E}$, it says that

$$\int_{\mathcal{E}} \phi(\eta)d\mu_N(u)(\eta) = \frac{1}{N}\sum_{i=1}^{N} \phi(u_i)$$

With this definition, the convergence for each continuous bounded test function f of $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ towards $\int f(\xi)d\mu(\xi)$ is exactly the narrow convergence of the sequence μ_{x_n} towards μ.

The set $\mathcal{P}(\mathbb{R}^d)$ of probability measure on $\mathbb{R}^d$ is a convex subset of the functional vector space $\mathcal{M}^1(\mathbb{R}^d)$ of bounded measures on $\mathbb{R}^d$. We endow $\mathcal{P}(\mathbb{R}^d)$ with the *narrow topology* for which $\mu_n \to \mu$ if and only if for all continuous and bounded test function $f \in \mathcal{C}^b(\mathbb{R}^d)$, $\int f d\mu_n \to \int f d\mu$. $\mathcal{P}(\mathbb{R}^d)$ is a Polish state for this topology.

So instead of considering the random variable x which takes its values in $\mathbf{R}^d$, we consider the random variable δ_ξ which takes its values in the Polish space $\mathcal{P}(\mathbf{R}^d)$. If (x_k) is an i.i.d. sequence in $\mathbf{R}^d$ with probability law μ, then δ_{x_i} is an i.i.d.=20 sequence in $\mathcal{P}(\mathbf{R}^d)$ and its empirical mean is just $\mu_{(x_1,\dots,x_n)}$ the empirical measure of an i.i.d. sample of size n. That means that=20 Cramer's theorem extension to Polish spaces may be applied. This theorem is known as *Sanov theorem.*

Let us first recall the definition of the relative entropy with respect to a probability measure μ on $\mathbf{R}^d$.

Definition 19 *Let μ be a probability measure on $\mathbf{R}^d$. We define a convex function $\nu \in \mathcal{P}(\mathbf{R}^d) \to I(\nu,\mu) \in \mathbf{R}$ by:*

$$\begin{cases} I(\nu,\mu) = \int \log \frac{d\nu}{d\mu}(\xi)d\nu(\xi) \\ I(\nu,\mu) = \infty \;\; else \end{cases} \tag{67}$$

This function is called the <u>*relative entropy*</u> *with respect to μ*

then we may state the Sanov theorem [21], [18]

Theorem 24 *The sequence of empirical measure μ_n which are associated to size n i.i.d. sample of a probability law on $\mathbf{R}^d$ satisfy a full LDP with the* <u>*relative entropy*</u> *with respect to μ as the rate function.*

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