Mean Field Theory for Random Recurrent Spiking Neural Networks

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Abstract—Recurrent spiking neural networks can provide biologically inspired model of robot controller. We study here the dynamics of large size randomly connected networks thanks to "mean field theory". Mean field theory allows to compute their dynamics under the assumption that the dynamics of individual neuronsare stochastically independent. We restrict ourselves to the simple case of homogeneous centered gaussian independent synaptic weights. First a theoretical study allows to derive the mean-field dynamics using a large deviation approach. This dynamics is characterized in function of an order parameter which is the normalized variance of the coupling. Then various applications are reviewed which show the applicative potentiality of the approach.

Keywords : Mean field theory, recurrent neural networks, dynamical systems, spiking neurons.

1. Introduction

Recurrent neural networks were introduced to improve biological plausibility of artificial neural networks as perceptrons since they display internal dynamics. They are useful to implement associative recall. The first models were endowed with symmetric connexion weights which induced relaxation dynamics and equilibrium states as in [8]. Asymmetric connexion weights were further introduced which enable to observe complex dynamics and chaotic attractors. The role of chaos in cognitive functions was first discussed by W.Freeman and C.Skarda in seminal papers as [11]. The practical importance of such dynamics is due to the use of on-line hebbian learning to store dynamical patterns. More recent advances along that direction are presented in the present conference [7].

The nature of the dynamics depends on the connexion weights. When considering large size neural networks, it is impossible to study the dynamics in function of the detailed parameters. One may consider that the connexion weights share few values, yet, the effect of the variablility cannot be studied by this approach. We consider here large random models where the connexion weights form a random sample of a probability law. These models are called *"Random Recurrent Neural Networks"*(RRNN). In that case, the parameters of interest are the *order parameters* i.e. the statistical parameters. Then the dynamics is amenable because one can approach it by *"Mean-Field Equations"* (MFE) as in Statistical Physics. MFE were introduced for neural networks by Amari [1] and Crisanti and Sompolinsky [12]. We extended their results [4] and used a new approach to prove it in a rigorous way [10]. This approach is the **"Large deviation Principle"** (LDP) and comes from the rigorous statistical mechanics [2]. We developped it for analog neuron model. We show here how it can be extended to spiking neural networks.

2. Random Recurrent Neural Networks

2.1. The neuron free dynamics

We consider here discrete time dynamics with finite horizon. The state of an individual neuron i at time t is described by the *membrane potential* $u_i(t) \in R$. For commodity, we shift it by the neuron firing threshold θ. So the trajectory of the potential of a single neuron is a vector of $\mathcal{F} = R^{\{0,1,\ldots,T\}}$. First let us consider the free dynamics of a neuron. We introduce $(w_i)(t))_{t\in\{1,\ldots,T\}}$ which is a sequence of i.i.d. centered Gaussian variables of variance σ^2. This sequence is called the *synaptic noise* of neuron i and stands for all the defects of the model; σ is an order parameter which is small. We shall consider three types of neuron: binary formal neuron (BF), analog formal neuron (AF) and integrate and fire neuron (IF). For BF and AF neuron, the free dynamics is given by the following equation

$$u_i(t+1) = w_i(t+1) - \theta \qquad (1)$$

For IF neuron, the free dynamics is given by

$$u_i(t+1) = \varphi[u_i(t) + \theta)] + w_i(t+1) - \theta \qquad (2)$$

where $\gamma \in]0,1[$ is the *leak* and where φ is defined by

$$\varphi(u) = \begin{cases} \gamma u \text{ if } \frac{\vartheta}{\gamma} < u < \theta \\ \vartheta \text{ else} \end{cases} \qquad (3)$$

ϑ is the reset potential and $\vartheta < 0 < \theta$. Let P be the distribution of the state trajectory of the neuron under the free dynamics. For a given initial distribution m_0, it is possible to explicit P for BF and AF neurons:

$$P = m_0 \times \mathcal{N}(-\theta, \sigma^2)^{\otimes T} \tag{4}$$

2.2. The synaptic potential of RRNN

To define the network dynamics, one has to introduce the activation variable $x_i(t)$ of the neuron at time t. For BF and IF neurons $x_i(t) = 1$ if and only if neuron i emits a spike at instant t, otherwise $x_i(t) = 0$. For AF neurons $x_i(t) \in [0, 1]$ represents the mean firing rate. In any case, is a non-linearfunction of $u_i(t)$ according to $x_i(t) = f[u_i(t)]$ where f is the *transfer function* of the neuron equal to the Heaviside function for BF and IF neurons and to the sigmoid function for AF neuron). Let us note $u = (u_i(t)) \in \mathcal{F}^N$ the network trajectory. The spikes are used to transmit information to other neurons through the synapses. Let us note $\mathcal{J} = (J_{ij})$ the system of *synaptic weights*. The *synaptic potential* of neuron i of a network of N neurons at time $t+1$ is a vector in $\mathcal{F}$ which is expressed in function of $\mathcal{J}$ and u by

$$\begin{cases} v_i(\mathcal{J}, u)(0) = 0 \\ v_i(\mathcal{J}, u)(t + 1) = \sum_{j=1}^{N} J_{ij} f[u_j(t)] \end{cases} \tag{5}$$

For size N RRNN model with gaussian connexion weights, $\mathcal{J}$ is a normal random vector with $\mathcal{N}(\frac{\overline{\upsilon}}{N}, \frac{\upsilon^2}{N})$ independent components. Notice that the RRNN model properties can be extended to a more general setting where the weights are non gaussian and depend on the neuron class in a several population model [5]

When u is given, $v_i(., u)$ is a gaussian vector in $\mathcal{F}$; its law is defined by its mean and its covariance matrix. Notice that these parameters depend only of the *empirical distribution* on $\mathcal{F}$ defined by $\mu_u = \sum_{i=1}^{N} \delta_{u_i} \in \mathcal{P}(\mathcal{F})$. They are invariant by any permutation of the neuron potential.

For $\mu \in \mathcal{P}(\mathcal{F})$ let us denote by g_μ the normal distribution on R^T with moments m_μ and c_μ:

$$\begin{cases} m_\mu(t + 1) = \overline{\upsilon} \int f[\eta(t)] d\mu(\eta) \\ c_\mu(s + 1, t + 1) = \upsilon^2 \int f[\eta(s)] f[\eta(t)] d\mu(\eta) \end{cases} \tag{6}$$

Proposition 1 *The common probability law of the individual synaptic potential trajectories $v_i(., u)$ is the normal law g_{μ_u} where μ_u is the empirical distribution of the network potential trajectory u.*

2.3. The network dynamics

Then the state of neuron i at time t is updated according to a modification of equation (2) for AF and BF models (resp. (3) for IF models) where the noise $w_i(t + 1)$ is replaced by $v_i(t + 1) + w_i(t + 1)$ fro each t. So gaussian vector computations lead to

Theorem 2 *Let $Q_N \in \mathcal{P}(\mathcal{F}^N)$ be the probability law the network potential trajectory for RRNN. Q_N is absolutely continuous with respect to the law $P^{\otimes N}$ of the free dynamics and $\frac{dQ_N}{dP^{\otimes N}}(u) = \exp N\Gamma(\mu_u)$ where the functional Γ is defined on $\mathcal{P}(\mathcal{F})$ by*

$$\Gamma(\mu) = \int \log\{\int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[\Phi_{t+1}(\eta)\xi(t) - \frac{1}{2}\xi(t)^2\right] dg_\mu(\xi)\} d\mu(\eta) \tag{7}$$

with

- *for AF and BF models: $\Phi_{t+1}(\eta) = \eta(t + 1) + \theta$*
- *IF model: $\Phi_{t+1}(\eta) = \eta(t + 1) + \theta - \varphi[\eta(t) + \theta]$*

The law of the empirical measure in the free model is just the law of an i.i.d. N-sample of P. An immediate consequence of the theorem is that $\exp N\Gamma(.)$ is the density of the law of the empirical measure in the RRNN model with respect to the law of the empirical measure in the free model.

3. The mean-field equation

3.1. The basis of LDP approach

Our objective is to compute the limit of the random measure μ_u when the size N of the network goes to infinity. By the Sanov theorem we know that in the free dynamics model μ_u satisfies a Large Deviation Principle (LDP) with the cross-entropy $I(\mu, P)$ as a good rate function and thus converges exponentially towards P. So by taking advantage of theorem 2, we are lead to the following statement

Large deviation principle *Under the law Q_N of the RRNN model μ_u satisfies a LDP principle with good rate function H defined by $H(\mu) = I(\mu, P) - \Gamma(\mu)$*

Actually, the rigorous proof is quite technical and some additonal hypothesis and approximations are necessary to follow the approach of [2]. The mathematical proof for AFRRNN is detailed in [10]. The keypoint is that for all RRNN models, it is possible to explicit the minimum of the rate function.

3.2. The mean-field propagation operator

Suppose that the u_i are iid according μ. Then, from the central limit theorem, the law of the v_i is g_μ in the limit of large networks. So if we feed a trajectory with a random synaptic potential distributed according g_μ, we obtain a new probability distribution on $\mathcal{F}$ which is noted $L(\mu)$.

Definition 1 *Let μ a probability law on $\mathcal{F}$ such that the law of the first component is m_0. Let u, w, v be three independent random vectors with the respective* *laws $\mu, \mathcal{N}(0, \sigma^2 I_T), g_\mu$ Then $L(\mu)$ is the probability law on $\mathcal{F}$ of the random vector ϑ which is defined by*

$$\left\{ \begin{array}{l} \vartheta(0) = u(0) \\ \vartheta(t+1) = v(t+1) + w(t+1) - \theta \end{array} \right. \quad (8)$$

for the formal neuron models (BF and AF), and by

$$\left\{ \begin{array}{l} \vartheta(0) = u(0) \\ \vartheta(t+1) = \varphi[u(t) + \theta)] + v(t+1) + w(t+1) - \theta \end{array} \right. \quad (9)$$

for the IF neuron model.The operator L on $\mathcal{P}(\mathcal{F})$ as defined above is called the mean-field propagation operator

Then we have

Proposition 3 *The density of $L(\mu)$ over P is*

$$\int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2}\xi(t+1)^2 + \Phi_{t+1}(\eta)\xi(t+1) \right] dg_\mu(\xi)$$

It is clear from the construction of L that $L^T(\mu) = \mu_0$ is a fixed point of L which depends only on the distribution m_0 of the inital state. From the previous proposition, we get

Theorem 4 *We have $I(\mu_0, P) = \Gamma(\mu_0)$ and so $H(\mu_0) = 0$*

Provided that μ_0 is the *only* minimum of H, this last theorem shows that the random sequence $(\mu_u)_N$ converges exponentially in law to μ_0 when $N \to \infty$

3.3. The main results of MFE theory

The independence of the (u_i) has been used to build the mean-field propagation operator but it cannot be checked exactly since the neuron states are correlated. The LDP principle allows to prove rigorously the *propagation of chaos* property. It amounts to the asymptotic independence of any *finite* set of individual trajectories.

Propagation of chaos property *Let h_1,...h_n be n continuous bounded test functions defined on $\mathcal{F}$, we have when $N \to \infty$*

$$\mathrm{E}[h_1(u_1)...h_n(u_n)] \to \prod_{i=1}^{n} \int h_i(\eta) d\mu_0(\eta) \quad (10)$$

An important consequence of the exponential convergence is the almost sure weak convergence. This result allows to use MFE for statements that relies on a single large network.

Theorem 5 *Let h be a continuous bounded test function defined on $\mathcal{F}$, we have when $N \to \infty$*

$$\frac{1}{N} \sum_{i=1}^{N} h(u_i) \to \int h(\eta) d\mu_0(\eta) \ \ as \quad (11)$$

Remark: The mathematical derivations of the previous results from LDP may be found in [10]. They are available for continuous test functions. For spiking neurons, the transfer function is not continuous, so we have to use a regular approximation of f to apply the previous theorems. Though this approximation cannnot be uniform, it is sufficient for the applications.

4. Applications to the dynamical regime of RRNN

The meann-field equations are used to predict the spontaneous dynamics of RRNN and to implement learning process on the "edge of chaos".

4.1. BFRRNN

For formal neurons, it is clear from (8) that $L(\mu)$ is gaussian. Moreover in the case of BFRRNN, the law of $L(\mu)(t+1)$ depends only on $\overline{x}_\mu(t) = \int f[\eta(t)]d\mu(\eta)$ which is the mean firing rate at time t. Thus, if we set $F(\theta) = \frac{1}{\sqrt{2\pi}} \int_\theta^\infty e^{-\frac{u^2}{2}} du$ we have

$$\overline{x}_{L(\mu)}(t+1) = F\left(\frac{\theta - \overline{\upsilon x}_\mu(t)}{\sqrt{\upsilon^2 \overline{x}_\mu(t) + \sigma^2}} \right) \quad (12)$$

So it is possible to get from the fixed point of that recurrence equation a bifurcation map. Three regimes appear a "dead one" where there is no firing, an intermediate one with a stable firing rate and a "saturated" one with a firing rate equal to 1. The dead regime and the saturated regime are aboslute if $\sigma = 0$. They tend to disappear when the variability of the connexion weights is increasing. Note that this approach is supposing the commutation of time limit and size limit since the mean-field theory was justified for finite-time horizon. Simulations ($N = 100$ are in a complete agreement with the theoretical predictions aand the stationary regime is reached within few time iterations.

4.2. AFRRNN

The results of the theory have been widely extended in [10]. First the hypothesis of gaussian connection can be dropped if it is replaced by an hypothesis of sub-gaussian tails for the distribution of the connexion weights. MFE can be written which describes the evolution of the empitical distribution of the network activity along time [4]. Thus, the distribution of the individual activities at a given time does not contain enough information about the nature of the dynamics. It may be stationary while the neuron states are stable or while the individual neurons describe synchronous or asynchronous trajectory. We are interested in the dynamic regime of the detailed network in the low-noise limit. A relevant quantity for that purpose is the evolution equation of the distance between two trajectories along time [6],[3]. Two initial states are selected independently and the dynamics are similar with the same configuration parameters and independent low noise. Then a mean-field theory is developped for the joint law of the two trajectories and allow to study the evolution of the mean-qudratic distance between the two trajectories in the low-noise limit. A limit equal to 0 is characteristic of a fixed point, a non-zero limit is the signature of the chaos. These informations may be recovered by the the study of the asymptotic covariance of the MFE for a sigle network.

For instance we applied this approach to predict the behaviour of a 2 population model (excitatory/ inhibitory) in [5]. It was studied in detail with the following order parameters (the index label the presynaptic and the postsynaptic population: g is quantifying the non-linearity of the transfer function of the AFRRNN, d is quantifying the inhibitory or excitatory character of the two populations. $d = 0$ amounts to the one-population model. Four asymptotic regimes exist: a stable fixed point regime, a stable periodic regim, a chaotic stationary regime and a cyclostationary chaotic regim. Simulation results and theoretical predictions were in good agreement.

4.3. IFRRNN

Mean-field theory is generally considered as a good approximation for IFRRNN [9]. Actually, the detailed model and the mean-field dynamics exhibit a transition from a zero mean-firing rate to a non-zero mean-firing rate when the standard deviation of the connexion weight is increasing. When the lack is growing to one, the critical standard deviation which induces a non)zero firing rate is growing. Still, there is a good quantitative agreement between simulations and theoretic MFE predictions. Yet, one is obliged to compute MFE predictions to use Monte-Carlo algorithms to simulate MFE dynamics. Another way of using the mean-field assumption to predict the firing-rate value is to model the synaptic potential as a random sum of independent random variables using Wald identites. It is developped in [9] and allows to predict the theoretical mean-firing rate without using Monte-Carlo simulations.

5. Perspectives

A general framework was proposed to study Random Recurrent Neural Networks using mean-field theory. It allows to predict simulation results for large size random recurrent neural network dynamics. IFRRNN models deserve further investigation. Notably, the model of random connections is far form biological models and random connectivity has to be tested.

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References

[1] S. Amari, K. Yosida, and K.I. Kakutani. A mathematical foundation for statistical neurodynamics. *Siam J. Appl. Math.*, 33(1):95–126, 1977.

[2] G. Ben Arous and A. Guionnet. Large deviations for langevin spin glass dynamics. *Probability Theory and Related Fields*, 102:455–509, 1995.

[3] B. Cessac. Increase in complexity in random neural networks. *Journal de Physique I*, 5:409–432, 1995.

[4] B. Cessac, B. Doyon, M. Quoy, and M. Samuelides. Mean-field equations, bifurcation map and route to chaos in discrete time neural networks. *Physica D*, 74:24–44, 1994.

[5] E. Daucé, O. Moynot, 0.Pinaud, and M. Samuelides. Mean field theory and synchronization in random recurrent neural networks. *Neural Processing Letters*, 14:115–126, 2001.

[6] B. Derrida and Y. Pommeau. *Europhys. lett.*, 1:45–59, 1986.

[7] E.Daucé, H.Soula, and G.Beslon. Learning methods for dynamic neural networks. In *NOLTA Conference, Bruges*, 2005.

[8] J. J. Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proc. Nat. Acad. Sci.*, 79:2554–2558, 1982.

[9] H.Soula, G.Beslon, and O.Mazet. Spontaneous dynamics of random recurrent spiking neural networks. *Neural Computation, accepted for publication*, 2005.

[10] O.Moynot and M.Samuelides. Large deviations and mean-field theory for asymetric random recurrent neural networks. *Probability Theory and Related Fields*, 123(1):41–75, 2002.

[11] C.A. Skarda and W.J. Freeman. *Chaos and the new science of the brain*, volume 1-2, pages 275–285. In "Concepts in Neuroscience", World Scientfic Publishing Company, 1990.

[12] H. Sompolinsky, A. Crisanti, and H.J. Sommers. Chaos in random neural networks. *Phys. Rev. Lett.*, 61:259–262, 1988.