State space

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State space is the set of all possible states of a dynamical system; each state of the system corresponds to a unique point in the state space. For example, the state of an idealized pendulum is uniquely defined by its angle and angular velocity, so the state space is the set of all possible pairs "(angle, velocity)", which form the cylinder $S^1 \times \mathbb{R}$, as in Fig.1.

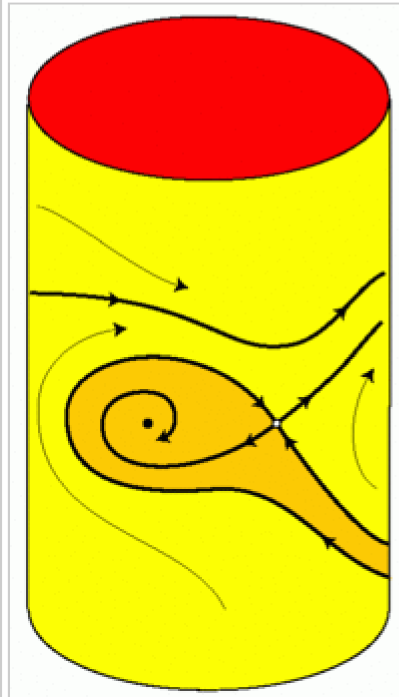

Figure 1: Phase portrait of a damped pendulum with a torque (see VCON).

In general, any abstract set could be a state space of some dynamical system. A state space could be *finite*, consisting of just a few points. It could be *finite-dimensional*, consisting of an infinite number of points forming a smooth manifold, as usually the case in ordinary differential equations and mappings. Such a state space is often called a **phase space**. A state space could be *infinite-dimensional*, as in partial differential equations and delay differential equations. In symbolic dynamics it is a Cantor set, which is *zero-dimensional*.

The number of **degrees of freedom** of a dynamical system is the dimension of its phase space, i.e., the number of variables the modeler feels is needed to completely describe the system. In the context of Hamiltonian systems, the number of degrees of freedom is the number of pairs of state variables.

Contents

Phase portrait

Dynamical regimes, such as a resting state or periodic oscillation, correspond to geometric objects, such as a point or a closed curve, in the phase space. Evolution of a dynamical system corresponds to a trajectory (or an orbit) in the phase space. Different initial states result in different trajectories. The set of of all trajectories forms the **phase**

portrait of a dynamical system, though in practice, only representative trajectories are considered. Since it is usually impossible to derive an explicit formula for the solution of a nonlinear equation, the analysis of phase portraits provides an extremely useful way for visualizing and understanding qualitative features of solutions.

Phase line

When the state of a dynamical system can be specified by a scalar value $x \in \mathbb{R}^1$ then the system is one-dimensional. Often, only a subset of the **phase line** $\mathbb{R}^1$ corresponds to physically meaningful states of the system, and it is often more natural to consider one-dimensional phase spaces in the form of intervals and circles. For example, the system could be a chemical reaction characterized by the concentration of a reagent or an RC-circuit characterized by the voltage across the capacitor. Notice that the former case, only non-negative values of $\mathbb{R}^1$ can be used, so the phase space is $[0, \infty)$.

One-dimensional systems are often given by the ordinary differential equation (ODE) of the form

$$x' = f(x),$$

where $x' = dx/dt$ is the derivative of the state variable x with respect to time t. This ODE is autonomous, i.e., f does not explicitly depend on the time t.

The phase line of a one-dimensional ODE is partitioned by the equilibria (points where $f(x) = 0$) and trajectories that connect the equilibria, as in Fig.2. The stability of the equilibria are determined by the directions of trajectories, which depend on the sign of the right-hand side function $f(x)$. One does not need to solve this equation, or even know the exact details of the function $f(x)$ to predict the dynamics of the system and its dependence on the initial condition; it is apparent from the phase portrait.

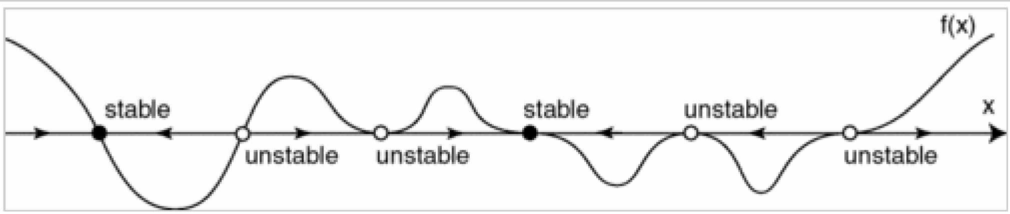

Figure 2: Phase portrait of $x' = f(x)$ depicts equilibria and typical trajectories on the the phase line $\mathbb{R}^1$ (reproduced from equilibrium).

One-dimensional systems can also be given by the iterated mapping in the form

$$x_{t+1} = f(x_t),$$

where the state at time $t + 1$ is a function of the state at time t. Phase portraits of such systems can be quite complicated, especially when the dynamics is chaotic. One-dimensional state spaces can also be more complicated, like graphs or dendrites.

Phase plane

Phase planes typically arise in the context of two-dimensional autonomous ODEs, which can be written in the form

$$x' = f(x,y)$$
$$y' = g(x,y).$$

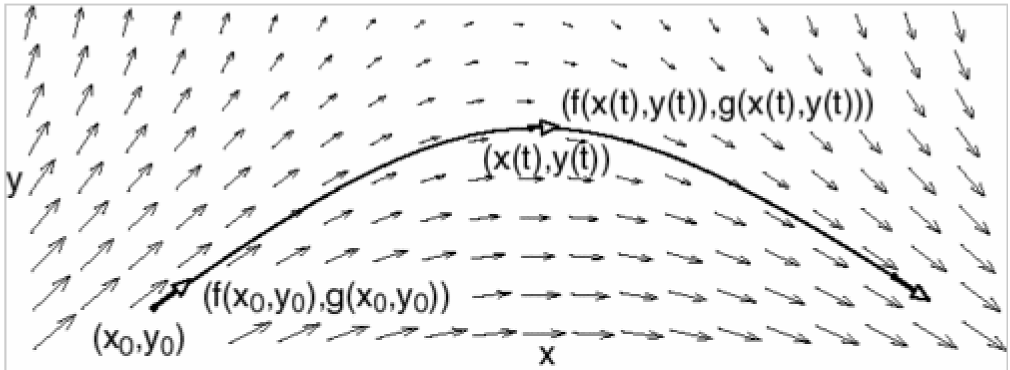

Figure 3: The phase plane. The right hand side of the two-dimensional dynamical system defined a vector field. Solutions of the equations define curves or trajectories in the phase plane. The vector field always points in the direction that the trajectories are flowing.

Here, f and g are given (smooth) functions. The two variables could describe, e.g., the position and velocity of a particle, the state of a predator-prey system, or concentrations of two reagents in a homogeneous chemical reaction.

If $(x(t), y(t))$ is a solution of the system, then at each time $t = p$, $(x(p), y(p))$ defines a point in the phase plane. The point changes with time, so the entire solution, $(x(t), y(t))$, traces out a curve, or **trajectory**, in the phase plane.

Of course, not every arbitrarily drawn curve in the phase plane represents a solution. What is special about solution trajectories is that the velocity vector at each point along the trajectory is given by the right hand side of the differential equation above. That is, the velocity vector of the trajectory $(x(t), y(t))$ at a point $(x(p), y(p))$ is given by $(x', y') = (f(x(p), y(p)), g(x(p), y(p)))$. This geometric property -- that the vector $(f(x, y), g(x, y))$ always points in the direction that the solution is flowing -- completely characterizes the solution trajectories (considered as subsets of the phase space). The function that assigns $(f(x, y), g(x, y))$ to (x, y) is called the **vector field**.

Equilibrium points of the two-dimensional dynamical system are where both $f = 0$ and $g = 0$. Note that if (x_0, y_0) is an equilibrium, then $(x(t), y(t)) \equiv (x_0, y_0)$ for all time is a (constant) solution of the system. Equilibria can be either stable or unstable.

A non-constant solution $(x(t), y(t))$ of a dynamical system is periodic if $(x(0), y(0)) = (x(T), y(T))$ for some $T > 0$. The minimal T that satisfies this requirement is called the **period**. Because $(x(t), y(t)) = (x(t+T), y(t+T))$ for all t, a periodic solution corresponds to a closed curve in the phase plane. Periodic solutions can be either stable or unstable. Roughly speaking, a periodic solution is stable if solutions that begin near the closed curve remain near for all $t > 0$ (this corresponds to orbital stability of a periodic orbit).

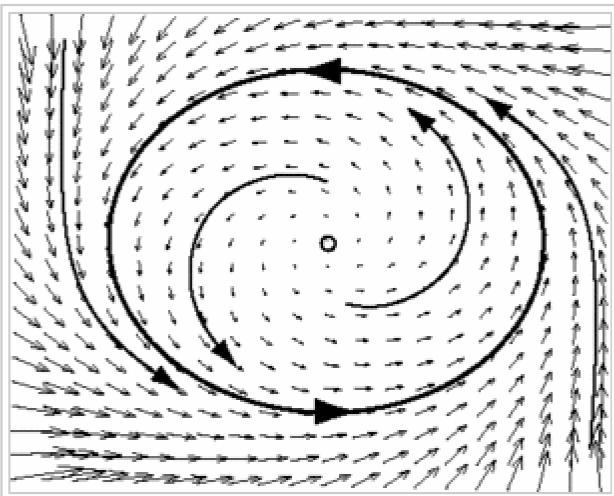
Figure 4: Periodic solutions correspond to closed curves in the phase plane.

It is usually much more difficult to locate periodic solutions than it is to locate equilibria. An equilibrium point (x_0, y_0) satisfies the equations $f(x_0, y_0) = g(x_0, y_0) = 0$ and these equations can usually be solved with straightforward numerical

methods. We also note that an equilibrium is a local
object -- it is simply one point in phase space. Oscillations or periodic orbits are global objects; they correspond to an entire curve in phase space that retraces itself. This curve may be quite complicated.

Two-dimensional phase spaces also arise in discrete dynamical systems of the form

$$x_{t+1} = f(x_t, y_t)$$
$$y_{t+1} = g(x_t, y_t).$$

In general, as in the continuous-time case, the phase space could be a subset of $\mathbb{R}^2$, or any surface with or without a boundary.

Higher dimensional systems

More generally, consider a system of n -first order differential equations of the form:

$$u' = F(u), \quad u \in R^n.$$

The phase space is simply n -dimensional Euclidean space and every solution, $u(t)$, corresponds to a trajectory in phase space parametrized by the independent variable t . As before, $F(u)$ defines a vector field in the phase space; at each point, $u(p)$, the vector $F(u(p))$ must be tangent to the solution curve $u(t)$. Moreover, equilibria are where $F(u) = 0$ and periodic solutions correspond to closed orbits.

Similarly, n -dimensional phase spaces arise in iterated mappings

$$u_{t+1} = F(u_t), \ u \in \mathbb{R}^n,$$

and it is often more natural to consider subsets of $\mathbb{R}^n$ or n -dimensional manifolds.

Abstract state spaces

According to the most abstract definition, a **dynamical system** *is homomorphism of an Abelian group (or semigroup) to a group of all automorphisms (endomorphisms in the case of semigroups) of a space* X . The space X is the state space of the dynamical system by definition. It could be any space with any topology or no topology at all; it could be finite or infinite. The abstract state spaces (with no structure) are typically considered by iteration theory, probability spaces by the ergodic theory, and topological spaces by dynamical systems theory, though often the former two theories are included into the *dynamical systems* theory. In applications, the choice of the state space should reflect the constrains of the system under consideration and must be as simple and intuitive as possible.

Examples

A simple phase portrait

Consider the system

$$\begin{aligned} x' &= y - x^2 + x \\ y' &= x - y. \end{aligned}$$

Note that there are two equilibria; these are at $(0,0)$ and $(2,2)$. Using the linearization method, we find that $(0,0)$ is a saddle and $(2,2)$ is a stable node. A useful way to analyze the phase plane is to draw the nullclines. The x -nullcline is where $x' = 0$; this is the curve $y = x^2 - x$. Along the x -nullcline, the vector field points either up or down, depending on the sign of y'. The y -nullcline is where $y' = 0$; this is the curve $y = x$. Along the y -nullcline, the vector field points to the left or to the right, depending on the sign of x'. Note that the nullclines divide the phase plane into separate regions; all of the vectors within a given region point towards the same quadrant. Once we have located the equilibria and have drawn the nullclines, it is usually possible to predict the behavior of the solution with some prescribed initial condition.

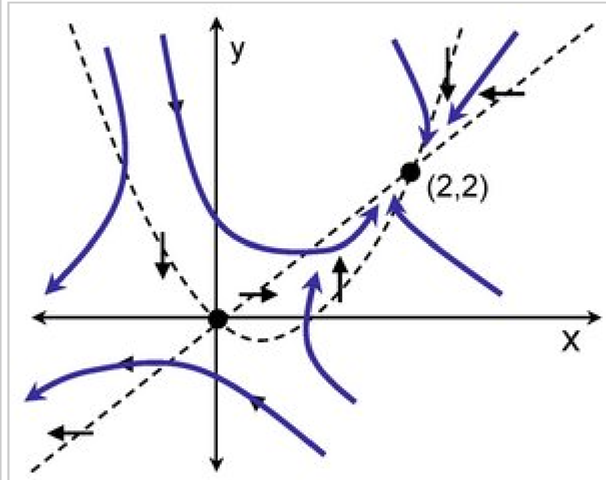

Figure 5: This example phase plane has equilibria at (0,0) and (2,2). The nullclines are shown with dashed curves and some trajectories are shown with solid curves.

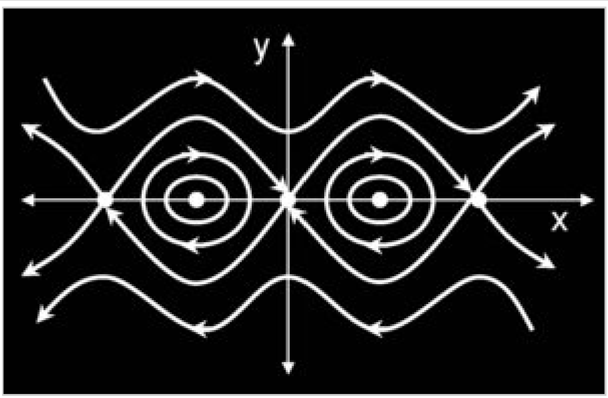

Figure 6: Phase portrait of the pendulum equation.

The pendulum

The equation for a pendulum can be written as $x'' + \sin(x) = 0$ where x is the angle from the downward vertical. In order to use phase plane analysis, we write this second order differential equation as the first order system:

$$\begin{aligned} x' &= y \\ y' &= -\sin(x). \end{aligned}$$

Note that there are infinitely many equilibria; these are at $(x,y) = (k\pi, 0)$ where k is any integer. There is no physical difference between angles that differ by 2π so we will only consider the equilibria at $(0,0)$ and $(\pi,0)$ (i.e., we consider the phase plane as the covering space for the cylinder). Using the linearization method, we find that $(0,\pi)$ is a saddle; the eigenvalues are ± 1. The eigenvalues at the origin are $\pm i$. Since these lie on the imaginary axis, we cannot determine the stability properties of the origin directly from the linearization method.

In order to draw the phase portrait, we use the fact that this system is conservative. That is, the total energy $E(x,y) = y^2/2 - \cos(x)$ is constant along solution trajectories. This can be easily verified by differentiating

$E(x(t), y(t))$ with respect to t and using the differential equation. Trajectories in the phase plane then represent curves of constant energy.

References

Internal references

- John Guckenheimer (2007) Bifurcation. Scholarpedia, 2(6):1517.
- James Meiss (2007) Dynamical systems. Scholarpedia, 2(2):1629.
- Eugene M. Izhikevich (2007) Equilibrium. Scholarpedia, 2(10):2014.
- James Meiss (2007) Hamiltonian systems. Scholarpedia, 2(8):1943.
- Kendall E. Atkinson (2007) Numerical analysis. Scholarpedia, 2(8):3163.
- Jeff Moehlis, Kresimir Josic, Eric T. Shea-Brown (2006) Periodic orbit. Scholarpedia, 1(7):1358.
- Philip Holmes and Eric T. Shea-Brown (2006) Stability. Scholarpedia, 1(10):1838.
- Emmanuil E. Shnol (2007) Stability of equilibria. Scholarpedia, 2(3):2770.

Recommended reading

- Izhikevich E. M. (2007) *Dynamical systems in neuroscience: The geometry of excitability and bursting*. The MIT Press.
- Strogatz, S. H. (1994). *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering*. Reading, MA: Perseus Books, Cambridge MA.
- Terman, D. (2005). An introduction to dynamical systems and neuronal dynamics, in *Tutorials in Mathematical Biosciences I, Mathematical Neuroscience*, Springer: 21-68.

External links

- David Terman's webpage (http://www.math.ohio-state.edu/~terman/)
- Eugene M. Izhikevich's webpage (http://www.izhikevich.com/)

See also

Attractors, Bifurcation, Dynamical system, Equilibrium, Periodic orbit, Relaxation oscillator, Stability, Stability of equilibria

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