Equilibrium

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An **equilibrium** (or **equilibrium point**) of a dynamical system generated by an autonomous system of ordinary differential equations (ODEs) is a solution that does not change with time. For example, each motionless pendulum position in Fig.1 corresponds to an equilibrium of the corresponding equations of motion, one is stable, the other one is not. Geometrically, equilibria are points in the system's phase space.

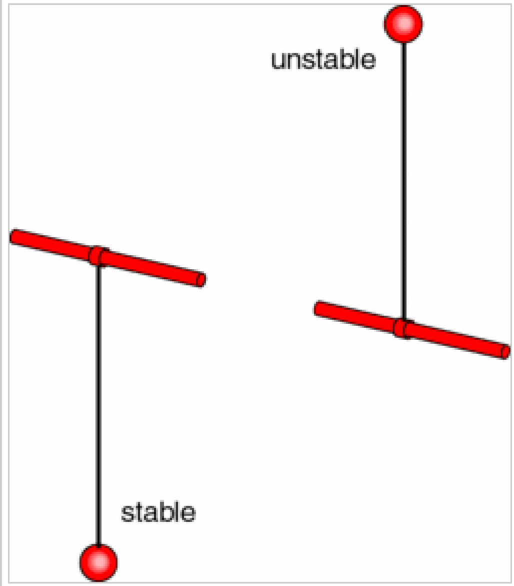


Figure 1: Illustration of a stable and unstable equilibrium point.

More precisely, the ODE

$$x' = f(x)$$

has an equilibrium solution $x(t) = x_e$ if $f(x_e) = 0$. Finding equilibria, i.e., solving the equation $f(x) = 0$ is easy only in a few special cases.

Equilibria are sometimes called fixed points or steady states. Most mathematicians refer to *equilibria* as time-independent solutions of ODEs, and to *fixed points* as time-independent solutions of iterated maps $x(t+1) = f(x(t))$.

Contents

Jacobian Matrix

The stability of typical equilibria of smooth ODEs is determined by the sign of real part of eigenvalues of the Jacobian matrix. These eigenvalues are often referred to as the 'eigenvalues of the equilibrium'. **The Jacobian Matrix** of a system of smooth ODEs is the matrix of the partial derivatives of the right-hand side with respect to state variables

$$J = D_x f = f_x = \left(\frac{\partial f_i}{\partial x_j}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \quad (1)$$

where all derivatives are evaluated at the equilibrium point $x = x_e$. Its eigenvalues determine linear stability properties of the equilibrium.

An equilibrium is **asymptotically stable** if all eigenvalues have negative real parts; it is **unstable** if at least one eigenvalue has positive real part.

Hyperbolic Equilibria

The equilibrium is said to be **hyperbolic** if all eigenvalues of the Jacobian matrix have non-zero real parts.

Hyperbolic equilibria are robust: Small perturbations of order ϵ, i.e., $x' = f(x) + \epsilon g(x, \epsilon)$, do not change qualitatively the phase portrait near the equilibria, but only displace equilibria by a small amount proportional to ϵ.

Moreover, local phase portrait of a hyperbolic equilibrium of a non-linear system is equivalent to that of its linearization. This statement has a mathematically precise form known as the Hartman-Grobman Theorem. It says that solutions of

$$x' = f(x)$$

in a small neighborhood of a hyperbolic equilibrium can be mapped with a homeomorphism (i.e., continuous map with a continuous inverse) onto solutions of the linear system

$$y' = Jy,$$

where J is the Jacobian matrix at the equilibrium. One says that these systems are locally topologically conjugate (equivalent). That is, adding nonlinear terms to a linear system at a hyperbolic equilibrium may distort but does not change qualitatively the phase portrait near the equilibrium.

If at least one eigenvalue of the Jacobian matrix is zero or has zero real part, then the equilibrium is said to be **non-hyperbolic**. Non-hyperbolic equilibria *are not* robust (i.e., the system is not structurally stable): Small perturbations can result in a local bifurcation of a non-hyperbolic equilibrium, i.e., it can change stability, disappear, or split into many equilibria. Some refer to such an equilibrium by the name of the bifurcation, e.g., *saddle-node equilibrium*.

In practice, one often has to consider non-hyperbolic equilibria with all eigenvalues having negative or zero real parts. These equilibria are sometimes referred to as being **critical**. Their stability cannot be determined from the signs of the eigenvalues of the Jacobian matrix; it depends on the nonlinear terms of f.

Types of Equilibria

One-Dimensional Space

Consider a one-dimensional (scalar) dynamical system

$$x' = f(x),$$
$$x \in \mathbb{R}^1$$

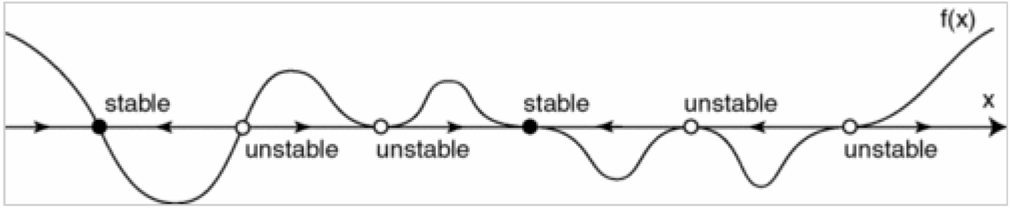


Figure 2: Equilibria of a one-dimensional system $x' = f(x)$ are the points where $f(x) = 0$.

with a differentiable (smooth) function $f(x)$. Its equilibria are the zeros of the function $f(x)$, as illustrated in Fig.2. The Jacobian matrix at each equilibrium is $J = f'(x)$. An equilibrium is asymptotically stable when $f'(x) < 0$; that is, the slope of f is negative. It is unstable when $f'(x) > 0$. The left two equilibria in the figure are hyperbolic ($f'(x) \neq 0$), the others are non-hyperbolic because the slope (eigenvalue) is zero. Nevertheless, a non-hyperbolic equilibrium of a one-dimensional system is stable if the function changes the sign from positive to negative at the equilibrium.

Two-Dimensional Space

Consider a two-dimensional (planar) system with smooth right-hand side

$$x_1' = f_1(x_1, x_2)$$
$$x_2' = f_2(x_1, x_2).$$

The Jacobian matrix has the form

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

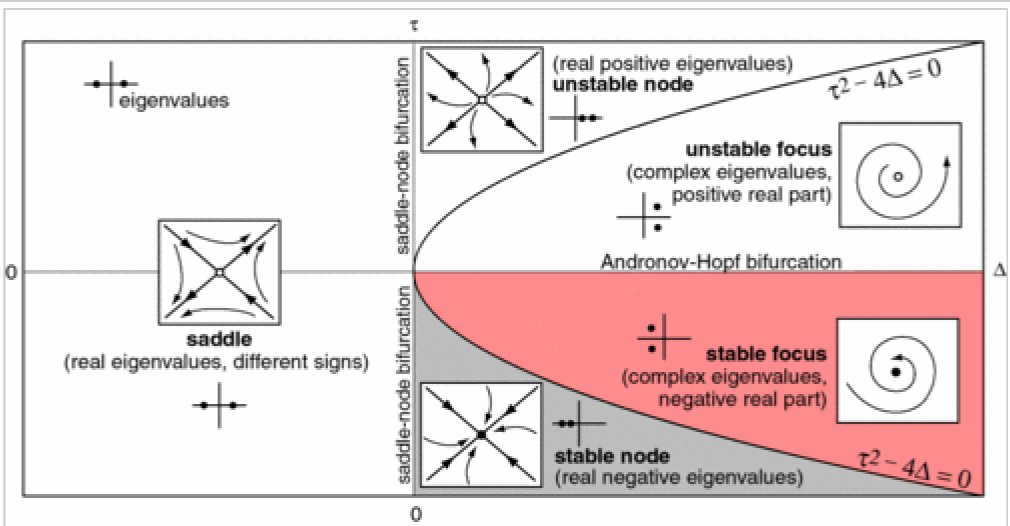


Figure 3: Classification of equilibria of a two-dimensional dynamical system according to the trace (τ) and the determinant (Δ) of the Jacobian matrix. The shaded region corresponds to stable equilibria. (modified from Izhikevich 2007).

It has two eigenvalues, which are either both real or complex-conjugate. A hyperbolic equilibrium can be a

- **Node** when both eigenvalues are real and of the same sign. The node is stable when the eigenvalues are negative and unstable when they are positive. For the stable node, the eigenvalue(s) with minimal absolute value of the real part is called **principle** or **leading**; when the eigenvalues are different, all orbits but two tend to the node along the leading eigenvector (the picture is reversed for the unstable node);

- **Saddle** when eigenvalues are real and of opposite signs. The saddle is always unstable;
- **Focus** (sometimes called **spiral point**) when eigenvalues are complex-conjugate; The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part.

Let

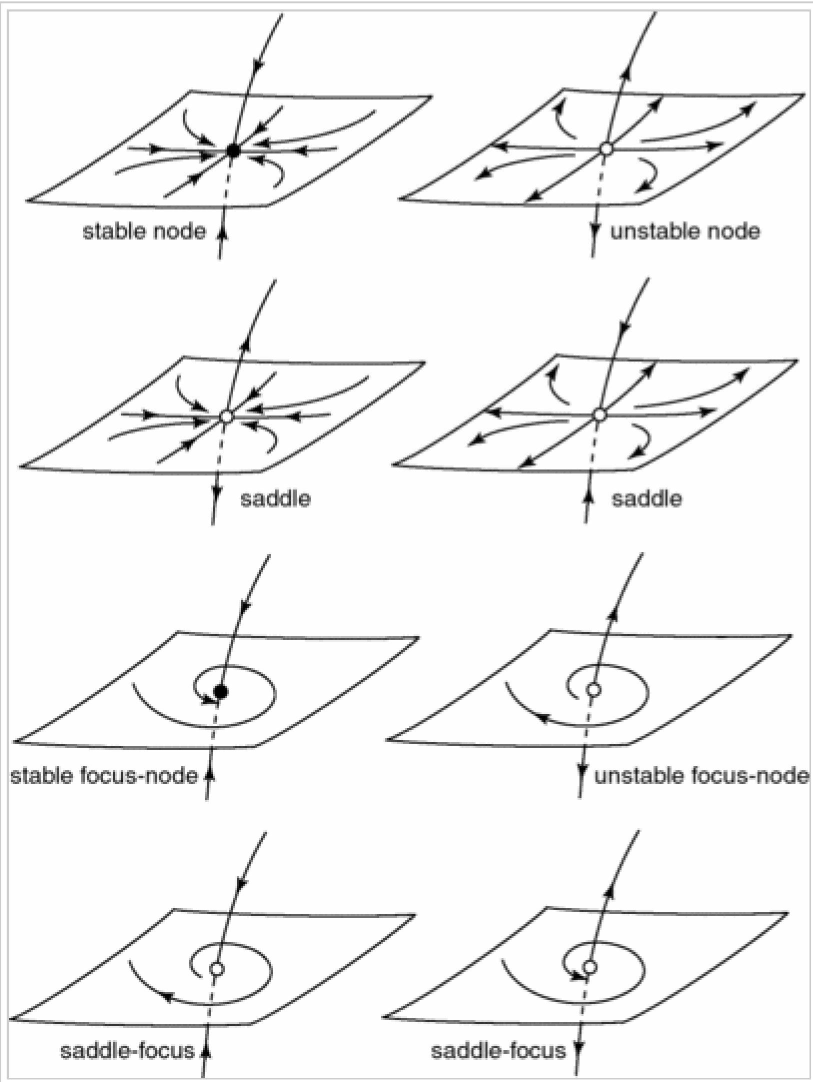
$$\tau = \mathrm{tr} J = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

be the *trace* and

$$\Delta = \det J = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}$$

be the *determinant* of the Jacobian matrix. Figure 3 summarizes the types of equilibria. The half-axis $\tau = 0, \Delta > 0$ and the axis $\Delta = 0$ correspond to nonhyperbolic equilibria that arise at Andronov-Hopf and Saddle-Node Bifurcation, respectively.

Three-Dimensional Space



The Jacobian matrix of a three-dimensional system has 3 eigenvalues, one of which must be real and the other two can be either both real or complex-conjugate. Depending on the types and signs of the eigenvalues, there are a few interesting cases illustrated in Fig.4. A hyperbolic equilibrium can be

- **Node** when all eigenvalues are real and have the same sign; The node is stable (unstable) when the eigenvalues are negative (positive);
- **Saddle** when all eigenvalues are real and at least one of them is positive and at least one is negative; Saddles are always unstable;
- **Focus-Node** when it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign; The equilibrium is stable (unstable) when the sign is negative (positive);
- **Saddle-Focus** when it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues; This type

of equilibrium is always unstable.

Figure 4: Examples of equilibria in $\mathbb{R}^3$.

Notice that nodes and focus-nodes change stability when time is reversed (i.e., when t is replaced by $-t$), whereas saddles and saddle-foci are unstable regardless of the direction of time.

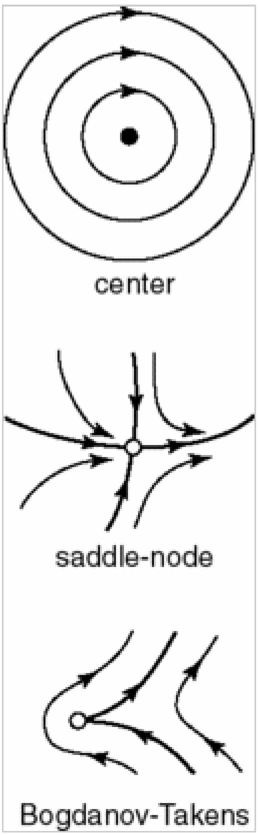


Figure 5: Examples of non-hyperbolic equilibria in $\mathbb{R}^2$. (The first and the third pictures correspond to Hamiltonian equations: $x'' = -\omega^2 x$ and $x'' = Kx^2$, $K > 0$, respectively.)

Nonhyperbolic Equilibria

There are many more types of non-hyperbolic equilibria, i.e., those that have at least one eigenvalue with zero real part, since the phase portrait in a small neighborhood of such equilibria also depends on the nonlinear terms of $f(x)$. Most of these equilibria do not have names or are named after the type of the bifurcation in which they play a role. Three examples are depicted in Fig.5.

The **center equilibrium** occurs when a system has only two eigenvalues on the imaginary axis, namely, one pair of pure-imaginary eigenvalues. Centers in linear systems have families of concentric periodic orbits around them, as in the figure. Many refer to centers only in the context of two-dimensional systems or Hamiltonian systems. If all other eigenvalues have negative real parts, centers are neutrally stable but not asymptotically stable. A pair of pure-imaginary eigenvalues also occurs in Andronov-Hopf bifurcation, however due to the nonlinear terms, the neighborhood of such an equilibrium looks like a focus; it could be asymptotically stable (supercritical Andronov-Hopf bifurcation) or unstable (subcritical Andronov-Hopf bifurcation) even if the other eigenvalues have negative real parts.
The **saddle-node equilibrium** occurs in nonlinear systems with one zero eigenvalue when the system undergoes the saddle-node bifurcation, where a saddle and a node approach each other, coalesce into a single equilibrium (depicted in the figure), and then disappear. Saddle-nodes are always unstable.
The **Bogdanov-Takens equilibrium** occurs in nonlinear systems with 2 zero eigenvalues, typically when the system undergoes the Bogdanov-Takens bifurcation. It is also an unstable equilibrium.

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See Also

Bifurcation, Dynamical Systems, Fixed Points, Hamiltonian Systems, Phase Space, Stability, Stability of Equilibria

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