# Reconsidering the Jeep Problem
## - Or How to Transport a Birthday Present to Salosauna -

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**Abstract**

The simple problem of how far a jeep can travel with a given amount of gasoline if intermediate gasoline dumps may be used is a nice example for problems which seem to have obvious recursive solution algorithms but which may become quite difficult if the problem specification is slightly changed. The classical version allows that arbitrarily small parts of the given amount of gasoline may be filled in the jeep's tank. Wood has restricted the problem to a discrete problem by requiring that the tank can be refilled only when it is empty and that it must be refilled completely, and that the gasoline is available only in cans of the size of the tank. In an earlier note we had shown, by using a new strategy, that the seemingly adequate algorithm given by Wood in analogy to the optimal solution to the classical version is not optimal. The new strategy however is also not optimal. In this note we discuss variants of the new strategy and try to get a better understanding of the influences of (small) changes in the problem specification to the solution algorithms.

### The Problem and Earlier Results

The classical jeep problem (see [Fin], [Phi], [Gal]) is to compute how far a jeep may go when starting at a dump with $n$ cans of gasoline, provided that it needs 1 canful to drive 1 unit distance and it is allowed to carry 2 canfuls of gasoline (including the contents of its tank). The optimal solution algorithm (see [Gal]) assumes that arbitrary fractions of a canful may be put into the tank.

Since this seems to be a bit unrealistic, a discrete variant was considered by D. Wood in [Woo]: It is only allowed to refill the tank, when it is empty and then 1 canful has to be filled in (thus 1 can can be transported). The following solution was proposed in [Woo]: With one canful transport $n-1$ cans to the next dump and go on recursively. This gives the distance function $f_w(n) = 1+1+\frac{1}{3}+\frac{1}{5}+\ldots+1/(2n-3)$. This seemed to be the natural adaptation of the classical optimal solution.

However in [Bra] we showed that $f_w(n)$ is not optimal for $n \geq 4$ by using a new strategy: With one canful transport only 1 can as far as possible. This means that, for even $n$, each second can is transported to a dump $\frac{1}{2}$ unit distance away - apart from the last one, which is moved to an auxiliary dump $\frac{3}{4}$ units away, and which is fetched in the next round with the last $\frac{1}{2}$ canful. This strategy should only be applied to even can numbers since if $n = 2k+1$ then only $k$ cans are used for the transportation of $k$ cans and the last one is wasted.

Let $[x;n]$ denote the situation, that at position $x$ there is a dump containing $n$ cans. We then can describe the procedure by

$$[0;2\cdot(2k-1)] \rightarrow [\frac{1}{2};2k-2]\&[\frac{3}{4};1] \rightarrow [1;k] \tag{1}$$

which means that the jeep starts at the dump at position 0 which contains $2\cdot(2k-1)$ cans. It then (in a first round) transports in each of $2k-2$ round trips (each using up one tankful) one can to a dump at position $\frac{1}{2}$, and in the final trip one can is moved to position $\frac{3}{4}$, and the jeep stops at position $\frac{1}{2}$ with an empty tank. In a second round $k-1$ cans are moved to a dump at position 1 in $k-1$ round trips and one trip from position $\frac{1}{2}$ to position 1 which leaves $\frac{1}{2}$ canful in the tank such that the jeep may bring the can from position $\frac{3}{4}$ to position 1. This strategy gives better results than Wood's strategy for $n = 6$ and all $n \geq 12$. The distance function for this procedure is

$$f_u(n) = f_u(p(t)) = (t+3)/2 \quad \text{for} \quad p(t) \le n < p(t+1) \quad \text{and} \quad t \ge 1.$$

$$\text{where} \quad p(t) = \begin{cases} (2^{t+1}+4)/3 = 2+2+2^3+\ldots+2^{t-1} & \text{for } t \text{ even} \\ (2^{t+1}+2)/3 = 2+2^2+2^4+\ldots+2^{t-1} & \text{for } t \text{ odd} \end{cases}$$

Its asymptotic behaviour is $\frac{1}{2}\lfloor log_2(3n)\rfloor + 1$ while that of $f_w(n)$ is $\frac{1}{2}ln\ n + 1.98$. When we compare the two strategies we are lead immediately to the assumption that the new strategy performs better because it needs less backward trips.

**Further Procedures**

The algorithms given in [Bra] are not always optimal. We still conjecture optimality for $n = p(t)$ where $t$ is odd. For other values of $n$ we have found several other (and better) solutions by slightly weakening the basic strategic idea of moving only 1 can as far as possible with 1 canful - such that it now seems that there is perhaps no universally optimal solution, but different procedures for different series of can numbers.

Procedure (1) suggests to consider the following sequences of can numbers $h_s(n)$ where $s$ is a (small) integer and $h_s(1) = s; h_s(n+1) = 4h_s(n) - 2, n \ge 1$. For example $s = 2$ gives the $p(t)$ where $t$ is odd. Obviously $f_u(h_s(n)) = f_u(s) + n - 1$ for $n \ge 2$.

To be able to work efficiently also with can numbers like 3, 4, 5, 6, 7, 8, 9, and also with the $p(t)$ for even $t$, we allow several auxiliary dumps at varying distances, and we move sometimes two cans forward by using the same tank filling; the resulting modified distance function is denoted by $f_{um}$.

$$[0;3] \to [\frac{1}{3};2] \quad \text{gives} \quad f_{um}(3) = 2\frac{1}{3}$$
$$[0;4] \to [\frac{1}{2}+\frac{1}{6};2] \quad \text{gives} \quad f_{um}(4) = 2\frac{2}{3}$$

Both these values are optimal, since they are the same as in the classical case.

$$[0;5] \to [\frac{1}{4};2]\&[\frac{5}{8};1] \to [\frac{5}{8}+\frac{5}{24};2], \quad \text{i.e. } f_{um}(5) = 2\frac{5}{6}.$$

This seems to be optimal also, since the distance for 5 cans in the classical case is $2\frac{13}{15}$.

More generally we obtain the following procedures:

$$[0;2\cdot 2n] \to [\frac{1}{2};2n-2]\&[\frac{1}{2}+\frac{1}{8};2], n \ge 2 \tag{2}$$

$$\begin{aligned} &[x;2j+2]\&[x+a;2] \to [x+\frac{1}{2};j]\&[x+\frac{1}{2}+\frac{1}{2}a;1], \&[x+a;2] \to \\ &\quad \to [x+\frac{1}{2};j]\&[x+\frac{1}{2}+\frac{1}{8}+\frac{1}{2}a;2], j \ge 2 \end{aligned} \tag{3}$$

$$\begin{aligned} &[x;4j+4]\&[x+a;2] \to [x+\frac{1}{2};2j+2]\&[x+\frac{4}{3}a;2] \to \\ &\quad \to [x+\frac{1}{2};2j+2]\&[x+\frac{3}{4}+\frac{2}{3}a;1] \to \\ &\quad \to [x+1;j]\&[x+1+\frac{4}{9}a;2], j \ge 2. \end{aligned} \tag{4}$$

$$[x;4]\&[x+a;2] \to [x+a;2]\&[x+\frac{1}{2}+\frac{1}{4}a;2] \tag{5}$$

Near the end of a jeep's tour we need the following procedure:

$$\begin{aligned}[x;2]\&[x+a;2] &\to [x+\frac{1}{2}+\frac{1}{2}a;1]\&[x+a;2]\to \\ &\to [x+\frac{1}{2}+\frac{1}{2}a+\frac{1}{3}(\frac{1}{2}+\frac{1}{2}a);2]=[x+\frac{2}{3}(1+a);2]\end{aligned} \tag{6}$$

For some odd $n$ we also found a (possibly optimal) starting procedure:

$$\begin{aligned}[0;7] &\to [\frac{1}{4};2]\&[\frac{1}{2}+\frac{1}{16};2] \\ [0;11] &\to [\frac{1}{4};6]\&[\frac{5}{8};1]\to[\frac{3}{4};2]\&[\frac{3}{4}+\frac{1}{16};2] \\ [0;4j+5] &\to [\frac{1}{4};2]\&[\frac{1}{2};2j]\&[\frac{1}{2}+\frac{1}{8};1]\to \\ &\to [\frac{1}{4}+\frac{1}{2};1]\&[\frac{1}{2}+\frac{1}{8}+\frac{1}{8};1]\&[\frac{1}{2};2j]= \\ &[\frac{1}{2};2j]\&[\frac{1}{2}+\frac{1}{4};2], j\geq 1.\end{aligned} \tag{7}$$

Applying these procedures in the appropriate order gives the following values of $f_{um}$:

$$\begin{aligned} f_{um}(7)=\frac{3}{18}, f_{um}(11) &= 3\frac{1}{3}+\frac{1}{8} \\ f_{um}(p(2j)) &= f_u(p(2j))+\frac{1}{16}\cdot(\frac{4}{9})^{j-2} \quad \text{for} \quad j\geq 2 \\ f_{um}(2^k) &= \frac{1}{2}k+\frac{3}{2}+\frac{1}{3}(1-2^{1-k}) \quad \text{for} \quad k\geq 1 \\ f_{um}(2^k+1) &= \frac{1}{2}k+\frac{3}{2}+\frac{1}{3} \quad \text{for} \quad k\geq 1\end{aligned}$$

**Discussion**

A problem arises when we consider the case $n = 10$. Obviously procedure (1) gives $[0;10]\to[1;3]$, and the optimal distance for $n = 3$ is $2\frac{1}{3}$. Thus it seems that the optimal distance for $n = 10$ should be $3\frac{1}{3} = f_u(h_3(2))$, since the jeep bringing 3 cans to position 1 by using procedure (1) always moves forward with a load of one can and never moves back behind a point where its tank was filled. Nevertheless already $f_{um}(9) = f_{um}(2^3+1) = 3\frac{1}{3}$. And indeed there is a better procedure for $n = 10$ by using procedures (5) and (6)

$$[0;10]\to[\frac{1}{4};4]\&[\frac{1}{2}+\frac{1}{16};2]\to[1\frac{1}{4}+\frac{5}{32};2]$$

That is, the distance reachable with $n = 10$ is $\frac{7}{96}$ units larger than that reached with procedure (1), although the jeep goes back (and forth) more often.

Therefore the main reason why Wood's strategy is not optimal, probably is not the high number of backward tours.

From this we also have to conclude that we cannot assume that the composition of good procedures gives again a good procedure.

Also the values $f_u(h_s(n))$ need not be optimal (although for example $f_u(14) = f_u(h_4(2)) = 3\frac{2}{3}$ looks quite well).

To get a better understanding of the solution possibilities, we may look at some natural constraints on possible solutions. We then see for example, that the constraint to have only one intermediate dump gives Wood's solution, while the constraint to have fixed distances between dumps implies to allow for at least 2 dumps; therefore if fixed distances and at most 2 dumps are to be used then procedure (1) is the only solution.

It remains open to find other reasonable sets of restrictions which determine useful strategies and to find a generally optimal algorithm.

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