Chapter 2

Random Recurrent Neural Networks Dynamics

(Manuel Samuelides, Bruno Cessac)

2.1 Introduction

Recurrent neural networks were introduced to improve biological plausibility of artificial neural networks as perceptrons since they display internal dynamics. They are useful to implement associative recall. The first models were endowed with symmetric connexion weights which induced relaxation dynamics and equilibrium states as in [21]. Asymmetric connexion weights were further introduced which enable to observe complex dynamics and chaotic attractors. The role of chaos in cognitive functions was first discussed by W.Freeman and C.Skarda in seminal papers as [30]. The practical importance of such dynamics is due to the use of on-line hebbian learning to store dynamical patterns. For a review see for instance [19]. More recent advances along that direction are presented in chapter 4.
In chapter 1, we were interested in the description of neural dynamics from the point of view of dynamical system theory. The present chapter is devoted to another kind of dynamical analysis, the so-called dynamic mean field theory. This method is inherited from statistical physics and has to be adapted to the present context.
From the point of view of dynamical system theory, the nature of the dynamics depends on the detailed configuration parameters such as connexion weights, firing thresholds, external input and so on.. When considering large size neural networks, it is impossible to study the dynamics in function of

the whole set of detailed configuration parameters because its dimensionality is too large. One may consider that the detailed parameters share few generic values but it does not allow to study the effect of their variability. We consider here random models where the detailed configuration parameters as connexion weights form a random sample of a probability distribution. These models are called *"Random Recurrent Neural Networks"*(RRNN). In that case, the parameters of interest define the probability distribution of the detailed configuration parameters. They are statistical parameters and have been introduced as "macroscopic parameters" in Chapter 1. Then the dynamics is amenable because one can approach it by *"Mean-Field Equations"* (MFE) as in Statistical Physics. So, study of the dynamics in terms of relevant dynamical quantities, called "*order parameters*" can be performed.

Mean-Field Equations were introduced for neural networks by Amari [2] and Crisanti and Sompolinsky [31]. We extended their results [10] and used a new approach to prove it in a rigorous way [27]. This approach is the **"Large deviation Principle"** (LDP) and comes from the rigorous statistical mechanics [5]. Simultaneously, mean-field equations were successfully used to predic the dynamics of spiking recurrent neural networks. Though there is no rigorous background to support these new developments, the success of this approach deserves new investigations. This chapter intends to provide a bridge between the detailed computation of the asymptotic regime and the rigorous theory which is shown to support in at least in some models MFE theory.
In section 2, the various models are stated from the points of view of the single neuron dynamics and of the global network dynamics. A summary of notations is presented, which is quite helpful for the sequel. In section 3 mean-field dynamics is developped and it is shown to be a fixed point of the mean-field propagation operator. The global dynamics probability distribution is computed and the associate empirical measure is proven to converge with exponential rate towards mean-field dynamics. The mathematical tools which are used there are detailed (without any proof) in appendix, section 6. In section 4, some applications of mean-field theory to the prediction of chaotic regime for analog formal random recurrent neural networks (AFRRNN) are displayed. The dynamical equation of homogeneous AFRRNN which is studied in chapter 1 is derived from the random network model in section 4.1. Moreover a two-population model is studied in section 4.2 and the occurence of a cyclo-stationary chaos is displayed using the results of [11]. In section 5, an insight of the application of mean-field theory to

IF networks is given using the results of [7]. The model of this section is a continuous-time model following the authors of the original paper. Hence the theoretical framework of the beginning of the chapter has to be enlarged to support this extension of mean-field theory and this work has to be done. However, we sketch a parallel between the two models to induce further research.

2.2 Dynamics of RRNNN

2.2.1 Defining dynamic state variables

We first reconsider in a stochastic context some notions that have been acquired in Chapter 1. In a stochastic model, all state variables may be considered as random variables.
We study here discrete time dynamics and restrain ourselves to finite time-horizon. In that section, we shall study the case of homogeneous neural networks. More realistic cases such as several population networks will be considered further on. Though we are really interested in long-time behaviour and stationary regime if any, rigorous proofs of convergence of large-size networks only exist for finite time. Thus we consider time t as an integer belonging to time interval $\{0, 1, ..., T\}$ where the integer T stands for the horizon.
Since the model is stochastic, all state variables may be considered as random variables. The change of probability distributions are common in that chapter whether it occurs from conditioning or from changing basic assumption over dynamics. Anyhow, we won't change notation for the random variables. Moreover following the standard physicist's habit, the random variables won't be noted by capital letters.
The state of an individual neuron i at time t is described by an instantaneous individual variable, the membrane potential $u_i(t)$. Here, $u_i(t)$ is a real random variable which takes its value in $\mathbb{R}$. So an individual state trajectory $u_i = (u_i(t))_{t\in\{0,1,...,T\}}$ takes its value in $\mathcal{F} = \mathbb{R}^{\{0,1,...,T\}}$. We shall generally prefer to study the distribution of a trajectory than the instantaneous state distribution. The second order moments of an individual trajectory u_i are its expectation $\mathrm{E}(u_i) \in \mathcal{F}$ and its covariance matrix $\mathrm{Cov}(u_i) \in \mathcal{F} \otimes \mathcal{F}$.
Our aim is to study the coupled dynamics of N interacting neurons that constitue a neural network. N is the **size** of the neural network. The global state trajectory of $u = (u_i)_{i\in\{1...,N\}}$ is a random vector in $\mathcal{F}^N$. The prob-

ability law[1] Q_N of the random vector u depends on N. We shall compute this law for various neuron models in this section.
As it was detailed in the previous chapter, the dynamics of the neuron models we study here depends on three basic assumptions

- about how the neuron activation depends on the membrane potential,
- about how the other neurons contribute to the synaptic potential which summarizes completely the influence of the network onto the target neuron,
- about how the synaptic potential is used to update the membrane potential.

We shall now detail these points in the models that are considered further on.

2.2.2 Spike emission modeling

It is considered that the neuron is active and emits a spike when its membrane potential exceeds the activation threshold. So the neuron i is active at time t when $u_i(t) \geq \theta$ where θ is the neuron **activation threshold**. We consider here that θ is a constant of the model which is the same for all neurons. Actually this hypothesis may be relaxed and random thresholds may be considered but the notation and the framework of dynamical study would be more complicate (see [27]).
For spiking neuron models we define an activation variable $x_i(t)$ which is equal to 1 if neuron i emits a spike at time t and to 0 otherwise. Hense we have

$$x_i(t) = f[u_i(t) - \theta] \tag{2.1}$$

where f is the Heaviside function which is called the **transfer function** of the neuron. Actually, to alleviate notations, we shift u_i of θ and that allows to replace equation (transfer0.eq) by equation

$$x_i(t) = f[u_i(t)] \tag{2.2}$$

The threshold will be further taken into account in the updating equation.
Two spiking neuron models are considered here, the **Binary Formal neuron (BF)** which is the original model of Mac Culloch aand Pitts [25] and

[1]This term is defined in Appendix, definition 2.7

the **Integrate and Fire neuron (IF)** which is generally used nowadays to model dynamics of large spiking neural networks [17].
In these models, the neuron activation takes generally two values: 0 and 1. This is true for most models of neurons. However, it was preferred in a lot of research works to take into account the average firing rate of the model instead of the detailed instant of firing. This point of view simplifies the model and deals with smooth functions which is easier from a mathematical point of view. According this point of view equation (2.2) is still valid but $x_i(t)$ takes its values in the interval $[0, 1]$ and the transfer function f is a smooth sigmoid function for instance $f(x) = \dfrac{e^x}{1+e^x}$. Since the activation of then euron is represented by a real vaue that varies continuously, the model is called **Analog Formal neuron (AF)**
AF model is still widely dominant when Artificial Neural Networks are considered for applications since gradient are easy to compute. For biological purpose, it was widely believed that the relevant information was stored in the firing rate; in that case more precise modeling would not be so useful, at least from a functional point of view. The three models are studied in that chapter and we attempt to give a unified presentation of mean-field equation for these three models.
Notice that the **state** of the neuron is the membrane potential $u_i(t)$ and not the activation $x_i(t)$. This is due to the update law of the IF model. We shall retun to that point further.

2.2.3 The synaptic potential of RRNN

The spikes are used to transmit information to other neurons through the synapses. Let us note $\mathcal{J} = (J_{ij})$ the system of **synaptic weights**. At time 0, the dynamic system is initialized and the synaptic potentials are null. The **synaptic potential** of neuron i of a network of N neurons at time $t+1$ is expressed in function of $\mathcal{J}$ and $u(t) \in \mathbb{R}^N$ by

$$v_i(\mathcal{J}, u)(t+1) = \sum_{j=1}^{N} J_{ij} x_j(t) = \sum_{j=1}^{N} J_{ij} f[u_j(t)] \tag{2.3}$$

When considering large size neural networks, it is impossible to study the dynamics in function of the detailed parameters. One may consider that the connexion weights share few values but it does not allow to study the effect of the variability. We consider here random models where the connexion weights form a random sample of a probability law. These models are called **"Random Recurrent Neural Networks"**(RRNN). In that case,

the parameters of interest are the **order parameters** i.e. the statistical parameters. For size N homogeneous RRNN model with gaussian connexion weights, $\mathcal{J}$ is a normal random vector with identically distributed independent components. The common law of the components is a normal law $\mathcal{N}(\frac{\overline{\upsilon}}{N}, \frac{\upsilon^2}{N})$. The order parameters of the model are $\overline{\upsilon}$ and υ.
Note that the assumption of independence is crucial in the approach described below. Unfortunately, the more realistic case where correlations between the Jijs exist (e.g. after Hebbian learning) is not currently covered by the mean-field methods. The description of the hebbian learning process in the stochastic RRNN framework has to be discovered in the future research.
In the sequel, we shall extend the RRNN model properties to a more general setting where the weights are non gaussian and depend on the neuron class in a several population model like in [11].
We have already dealt with the dynamical properties of RRRN such as (1.3). In Chapter 1, we fixed a realization of $\mathcal{J}$ and considered the evolution of trajectories of this dynamical system. Then, we averaged over $\mathcal{J}$ distribution in order to get informations about the evolution of averaged quantities. In the present chapter we shall start with a complementary point of view. Namely, assume that we fix a network state trajectory u. Let us consider the vector $v_i(\mathcal{J}, u) = (v_i(\mathcal{J}, u)(t)) \in \mathcal{F}$ which is the trajectory of the synaptic potential. When u is given, since the synaptic weights J_{ij} are gaussian, identically distributed and independant we infer that the $v_i(., u)$ are identically distributed normal independent random vectors in $\mathcal{F}$ (see in appendix proposition (2.17). The distribution of the v_i is defined by its mean m_u and its covariance matrix c_u.
We have

$$m_u(t+1) = \frac{\overline{\upsilon}}{N} \sum_{j=1}^{N} f[(u_j(t)] \tag{2.4}$$

and

$$c_u(s+1, t+1) = \frac{\upsilon}{N} \sum_{j=1}^{N} f[(u_j(s)] f[(u_j(t)] \tag{2.5}$$

Notice that these parameters are invariant by any permutation of the neuron membrane potentials. Actually, they depend only on the **empirical distribution**[2] μ_u, which is associated to u. μ is a probability law on $\mathcal{F}$. It just weights equally each individual neuron state trajectory.

[2]This concept is introduced in the appendix in definition (2.18)

Definition 2.1 *The empirical measure μ_u is an application from $\mathcal{F}^N$ on $\mathcal{P}(F)$, the set of probability measures on $\mathcal{F}$ which is defined by*

$$\mu_u(A) = \sum_{i=1}^{N} \delta_{u_i}(A) \tag{2.6}$$

where $\delta_{u_i}(A)$ is the Dirac mass on individual trajectory u_i such that $\delta_{u_i}(A) = 1$ if u_i belongs to A and 0 otherwise. Using this formalism provides an useful way to perform an average over the empirical distribution of the whole network. More generally, assume that we are given a probability law μ on the indivudual trajectory space $\mathcal{F}$. Then, one can perform a generic construction of a gaussian distribution on R^T by

Definition 2.2 *For any $\mu \in \mathcal{P}(\mathcal{F})$ the gaussian probability law g_μ on R^T with moments m_μ and c_μ that are defined by :*

$$\begin{cases} m_\mu(t+1) = \overline{v} \int f[\eta(t)] d\mu_u(\eta) \\ c_\mu(s+1, t+1) = v^2 \int f[\eta(s)] f[\eta(t)] d\mu_u(\eta) \end{cases} \tag{2.7}$$

Then, it is easy to reformulate the previous computation as:

Proposition 2.1 *The common probability law of the individual synaptic potential trajectories $v_i(., u)$ is the normal law g_{μ_u} where μ_u is the empirical distribution of the network potential trajectory u.*

Proposition 2.2 *The common probability distribution of the individual synaptic potential trajectories $v_i(., u)$ is the normal law g_{μ_u} where μ_u is the empirical distribution of the network potential trajectory u.*

This framework is useful to compute the large-size limit of the common probability law of the potential trajectories.

2.2.4 Dynamical models of the membrane potential

We shall now detail the updating rule of the membrane potential. Various neural dynamics have been detailed in thep revious chapter. We focus here on two dynamics formal neuron (AF and BF) and Integrate and Fire neuron (IF).
For the two models, the network is initialized with indepedent identically distributed membrane potential according to a probability law $\mu_{init} \in \mathcal{P}(\mathbb{R})$. We introduce for each neuron i, a sequence $(w_i)(t))_{t\in\{1,...,T\}}$ of i.i.d. centered Gaussian variables of variance σ^2. This sequence is called the **synaptic**

noise of neuron i and stands for all the defects of the model. The synaptic noise plays an important part in the mathematical proof but the order parameter σ is as small as necessary. So this model is not very restrictive. The synaptic noise is added to the synaptic potential. Of course the synaptic noises of the neurons are independent altogether. In some papers, the synaptic noise is called **thermal noise** by comparison with the random variables $\mathcal{J} = (J_{ij})$, which are called **quenched variables** as they are fixed, once for all and do not change with time.
Recall that formal neuron updates its membrane potential according to

$$u_i(t+1) = v_i(t+1) + w_i(t+1) - \theta \tag{2.8}$$

IF neuron takes into acccount its present membrane potential while updating. Its evolution equation is

$$u_i(t+1) = \varphi[u_i(t) + \theta)] + v_i(t+1) + w_i(t+1) - \theta \tag{2.9}$$

where

- $\gamma \in]0,1[$ is the **leak**
- φ is defined by
$$\varphi(u) = \begin{cases} \gamma u \text{ if } \frac{\vartheta}{\gamma} < u < \theta \\ \vartheta \text{ else} \end{cases} \tag{2.10}$$
- ϑ is the reset potential and $\vartheta < 0 < \theta$

The following table summarizes the main properties of the three models we investigate:

Transfer function	Heaviside	sigmoidal
Formal model	BF	AF
Integrate and Fire	IF	

If we shunt the neural coupling, then the individual neuron state trajectories are independent identically distributed random vectors in $\mathcal{F}$ and we note P their common law. The dynamics of the neural network state when the neural coupling is shunted is called the **free dynamics**. The probability law of the neural network trajectory is $P^{\otimes N}$.
In the case of formal neurons, the free dynamics equation is

$$u_i(0) \sim \mu_{init}, \quad u_i(t+1) = w_i(t+1) - \theta \tag{2.11}$$

and so $P = \mu_{init} \otimes \mathcal{N}(-\theta, \sigma^2)^{\otimes T}$. In the case of IF neurons P is not explicit. It is the image of $\mu_{init} \otimes \mathcal{N}(-\theta, \sigma^2)^{\otimes T}$ by the diffusive dynamics

$$u_i(0) \sim \mu_{init}, \quad u_i(t+1) = \varphi[u_i(t) + \theta)] + w_i(t+1) - \theta \tag{2.12}$$

When coupling the neurons, the network trajectory is still a random vector. Its probability distribution, which is denoted by Q_N, has a density with respect to $P^{\otimes N}$ that can be explicitely computed. This is the main topic of the next subsection.

2.2.5 Computation of the network dynamics law

This section is dedicated to the computation of the probability distribution Q_N. The result shows that the density of Q_N with respect to free dynamics probability $P^{\otimes N}$ depends on the trajectory variable u only through the empirical measure μ_u. To achieve this computation we shall use a key result of stochastic process theory, the Girsanov theorem which gives the density of the new law of a diffusion when the trend is changed. Actually, since the time set is finite, the version of Girsanov theorem we use is different and may be recovered by elementary gaussian computation. Its derivation is detailed in the appendix in theorem 2.20. A similar esult may be obtained for continuous time dynamics using the classical Girsanov theorem (see [5]). Let us state the finite-time Girsanov theorem

Theorem 2.3 *Let μ_{init} a probability measure on $\mathbb{R}^d$ and let $\mathcal{N}(\alpha, K)$ be a gaussian regular probability on $\mathbb{R}^d$. Let T a postive integer and $\mathcal{E}_T = (\mathbb{R}^d)^{\{0,\dots,T\}}$ the space of finite time traajectories in $\mathbb{R}^d$. Let w a gaussian random vector in $\mathcal{E}_T$ with law $\mu_{init} \otimes \mathcal{N}(\alpha, K)^T$. Let ϕ and ψ two measurable applications of $\mathbb{R}^d$ into $\mathbb{R}^d$. Then we define the random vectors x and y in $\mathcal{E}$ by*

$$\begin{cases} x_0 = w_0 \\ x(t+1) = \phi[x(t)] + w(t+1) \end{cases}$$

$$\begin{cases} y_0 = w_0 \\ y(t+1) = \psi[y(t)] + w(t+1) \end{cases}$$

Let P and Q be the respective probability laws on $\mathcal{E}$ of x and y, then Q is absolutely continuous with respect to P and we have

$$\frac{dQ}{dP}(\eta) = \exp \sum_{t=0}^{T-1} \left\{ \begin{array}{l} -\frac{1}{2}\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1} \{\psi[(\eta(t)] - \phi[(\eta(t)]\} \\ +\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1} \{\eta(t+1) - \alpha - \phi[\eta(t)]\} \end{array} \right\} \tag{2.13}$$

We shall use this theorem to prove the following:

Theorem 2.4 *The density of the law of the network membrane potential Q_N with respect to $P^{\otimes N}$ is given by*

$$\frac{dQ_N}{dP^{\otimes N}}(u) = \exp N\Gamma(\mu_u)$$

where the functionnal Γ is defined on $\mathcal{P}(\mathcal{F})$ by

$$\Gamma(\mu) = \int \log\left\{\int \exp \tfrac{1}{\sigma^2}\sum_{t=0}^{T-1}\left[-\tfrac{1}{2}\xi(t+1)^2 + \Phi_{t+1}(\eta)\xi(t+1)\right] dg_\mu(\xi)\right\} d\mu(\eta) \tag{2.14}$$

with

- *for AF and BF models:* $\Phi_{t+1}(\eta) = \eta(t+1) + \theta$
- *IF model:* $\Phi_{t+1}(\eta) = \eta(t+1) + \theta - \varphi[\eta(t) + \theta]$

Remark 2.1 *Let us recall that the gaussian measure g_μ has been defined previously (Definition 2.2)*

PROOF OF THEOREM: We note $Q_N(\mathcal{J})$ the conditional law of the network state trajectory given $\mathcal{J}$ the system of synaptic weights. We shall apply the finite-time Girsanov theorem 2.3 to express $\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}$. To apply the theorem we notice that

- The difference of the two trend terms $\psi[\eta(t) - \phi[\eta(t)]$ of the theorem is here the synaptic potentials $(v_i(t))$. The synaptic potentials v_i are functions of the u_i according to (2.3)

$$v_i(\mathcal{J}, u)(t+1) = \sum_{j=1}^{N} J_{ij} f[u_j(t)]$$

- The expression of the synaptic noise $(w_i(t+1)$ in function of the state trajectory $u_i$in the free dyamics is $\Phi_{t+1}(u_i)$ which expression depends on the neuron model (formal or IF).

We have so

$$\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}(u) = \exp \frac{1}{\sigma^2}\sum_{t=0}^{T-1}\sum_{i=1}^{N}\left[-\frac{1}{2}v_i(\mathcal{J}, u)(t+1)^2 + v_i(\mathcal{J}, u)(t+1)\Phi_{t+1}(u_i)\right] \tag{2.15}$$

$$\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}(u) = \prod_{i=1}^{N} \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2} v_i(\mathcal{J}, u)(t+1)^2 + v_i(\mathcal{J}, u)(t+1)\Phi_{t+1}(u_i) \right] \tag{2.16}$$

Now let us consider the probability of the quenched variables $\mathcal{J} = (\mathcal{J}_i j)$. We observed previously when we introduced the synaptic potential model that under the configuration distribution of $\mathcal{J}$, the random vectors $v_i(\mathcal{J}, u)$ are independent identically distributed according to the normal law g_{μ_u}
To compute the density of Q_N with respect to $P^{\otimes N}$ one has to average the conditional density $\frac{dQ_N(\mathcal{J})}{dP^{\otimes N}}$ over the configuration distribution of $\mathcal{J}$. the integration separates into products and one gets from (2.16) the following

$$\frac{dQ_N}{dP^{\otimes N}}(u) = \prod_{i=1}^{N} \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2} \xi(t)^2 + \Phi_{t+1}(u_i)\xi(t) \right] dg_{\mu_u}(\xi)$$

$$\frac{dQ_N}{dP^{\otimes N}}(u) = \exp \sum_{i=1}^{N} \log \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2} \xi(t+1)^2 + \Phi_{t+1}(u_i)\xi(t+1) \right] dg_{\mu_u}(\xi)$$

The sum over i is equivalent to an inegration over the empirical measure μ_u, so we have

$$\frac{dQ_N}{dP^{\otimes N}}(u) = \exp N \int \log \left\{ \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2} \xi(t+1)^2 + \Phi_{t+1}(\eta)\xi(t+1) \right] dg_{\mu_u}(\xi) \right\} d\mu_u(\eta)$$

■

Remark 2.2

Equation 2.14 reminds the generating functional approach derived e.g. by Sompolinsky al. [31] or Molgedey al [24] which allows to compute the moments of $u_i(t)$. However, the present approach provides a stronger result. While the generating functional method deals with weak convergence (convergence of generating function) the method that is developped here allows direct access to $u_i(t)$s probability distribution. Moreover, using large deviation techniques provides almost sure convergence results. This convergence is valid for only one typical sample. This property is also called *self-averaging* in the statistical physics community. Let us now state an important corollary of this theorem which will be the basic statement for the convergence theorem of next section.

Corollary 2.5 *The density of the law of the empirical measure μ_u as a random measure in the RRNN model that is governed by Q_N with respect to the law of the empirical measure in the free model that is governed by $P^{\otimes N}$ is*

$$\mu \in \mathcal{P}(\mathcal{F}) \to \exp N\Gamma(\mu)$$

2.2.6 Summary of notations

Let us recall the notations of this section. They will be extensively used in the following sections:

Notation	Interpretation
$i \in \{1, ..., N\}$	individual neuron of a N neuron population
$t \in \{0, ..., T\}$	time course of the discrete time dynamics at horizon T
$u_i(t)$	membrane potential of neuron i at time t
$x_i(t)$	activation state of neuron i at time t
$v_i(t)$	synaptic potential of neuron i at time t
$w_i(t)$	synaptic summation noise of neuron i at time t
$u_i \in \mathcal{F}$	membrane potential trajectory of neuron i from time 0 to time T
$u \in \mathcal{F}^N$	network membrane potential trajectory i from time 0 to time T
$x_i \in \mathcal{E}$	activation state trajectory of neuron i from time 0 to time T
$x \in \mathcal{F}^N$	network activation state trajectory from time 0 to time T
θ	common firing threshold of individual neurons
σ	common standard deviation of the synaptic noise
λ	leak current factor for Integrate and fire (IF) neuron model
f	neuron transfer function converts membrane potential into activation state
J_{ij}	synaptic weight from neuron j to neuron i (real random variable)
$\mathcal{J} = (J_{ij})$	synaptic weight matrix (random $N \times N$ matrix)
$\frac{\overline{v}}{N}$	expectation of synaptic weights J_{ij}
$\frac{v^2}{N}$	variance of synaptic weights J_{ij}
$\mu \in \mathcal{P}(\mathcal{F})$	generic probability law of individual membrane potential trajectory
$\eta \in \mathcal{F}$	random vector which takes its values in $\mathcal{F}$ under probability law μ
$P \in \mathcal{P}(\mathcal{F})$	probability law of individual membrane potential trajectory for free dynamics
$g_\mu \in \mathcal{P}(\mathcal{F})$	synaptic potential law obtained from $\mu \in \mathcal{P}(F)$ through central limit approximation
$Q_N \in \mathcal{P}(\mathcal{F}^N)$	probability law of network membrane potential trajectory u

2.3 The mean-field dynamics

2.3.1 Introduction to mean-field theory

The aim of this section is to describe in the limit of large size networks the evolution of a typical neuron by summarizing in a single term the effect of the interactions of this neuron with the other neurons of the network. Such an approximation will be valid through an averaging procedure which will take advantage of the large number of coupling to postone the vanishing of individual correlations between neurons or between neurons and configuration variables. This is the hypothesis of "local chaos" of Amari ([1],[2]), or of "vanishing correlations" which is usually invoked to support mean-field equations. The "mean-field" is properly the average effet of all the interaction of other neurons with the neuron of interest. This approach is currently used in statistical mechanics since Boltzmann and the assumption is also kown under "molecular chaos".[3]
So the mean field dynamics is intermediate between the **detailed** dynamics which takes into account all the detailed interactions between neurons and the free dynamics which neglects all the interaction. In mean-field theory, we are looking for an evolution equation of the type of free dynamics equation that are involving single neuron dynamics but were the interaction term is not cancelled. To do that we shall replace $v_i(t)$ by an approximation which depends only on the statistical distribution of the $u_j(t)$. The approximation of $v_i = (v_i(t)) \in \mathcal{F}$ will be called the **mean-field** and will be noted $\zeta = (\zeta(t)) \in \mathcal{F}$. The important assumption to derive the anzats is:
In the large size limit, u_j are asymptotically independent, they are also independent from the configuration parameters
From the central limit theorem we are then allowed to postpone that ζ is approximatively a large sum of independent identically distributed variable, and thus that it follows approximatively a Gaussian law. We have just to derive its first and second order moment from the common probability law of $u_i = (u_i(t))$ to know completely the distribution of ζ.
Thus from a probability law on $\mathcal{F}$ which is supposed to be the common

[3]The word "chaos" is somehow confusing here, especially because we also dealt with deterministic chaos in the first chapter. Actually, "deterministic chaos" and the related exponential correlation decay is connected to the mean-field approaches which allows to compute deterministic evolution equation for the mean value and Gaussian fluctuations. However, here the mean-field approach works basically because the model is fully connected and the J_{ij} are vanishing in the large-size limit. This is standard result in statistical physics models such as the Curie-Weiss model but obtaining this for the trajectories of a dynamical model with quenched disorder requires more elaborated technics.

probability law of the u_i, we are able to derive the law of ζ and then the law of the resulting potential trajectory and of the state trajectory of a generic vector of the network. In that way, we are able to define an evolution operator L on the set $\mathcal{P}(F)$ of the probability laws on $\mathcal{F}$ which we call the **mean-field propagation** operator.

2.3.2 Mean-field propagation and mean-field equation

Let $\mu \in \mathcal{P}(\mathcal{F})$ be a probability measure on $\mathcal{F}$ and let us compute the moments of

$$\forall t \in \{0, 1, ..., T-1\}, \zeta(t+1) = \sum_{j=1}^{N} J_{ij} f[u_j(t)]$$

where the u_j are independent identically distributed random vectors with probability law μ and independent from the configuration parameters J_{ij}. Since $E[J_{ij}] = \frac{\overline{v}}{N}$ and $\mathrm{Var}[J_{ij}] = \frac{v^2}{N}$, we have

$$\begin{cases} E[\zeta(t+1)] = \overline{v} \int_{\mathcal{F}} f[\eta(t)] d\mu(\eta) \\ \mathrm{Cov}[\zeta(s+1), \zeta(t+1)] = v^2 \int_{\mathcal{E}} f[\eta(s)] f[\eta(t)] d\mu(\eta) \end{cases} \tag{2.17}$$

Notice that the expression of the covariance is asymptotic since the sum of squares of expectation of the synaptics weights may be neglected. So ζ is a Gaussian random vector in $\mathcal{F}$ with probability law g_μ (see definition 2.2).

Definition 2.3 *Let μ a probability law on $\mathcal{F}$ such that the law of the first component is μ_{init}. Let u, w, v be three independent random vectors with the following laws*

- *the law of u is μ,*
- *the law of w is $\mathcal{N}(0, \sigma^2 I_T)$,*
- *the law of v is g_μ*

Then $L(\mu)$ is the probability law on $\mathcal{F}$ of the random vector ϑ which is defined by

$$\begin{cases} \vartheta(0) = u(0) \\ \vartheta(t+1) = v(t+1) + w(t+1) - \theta \end{cases} \tag{2.18}$$

for the formal neuron models (BF and AF), and by

$$\begin{cases} \vartheta(0) = u(0) \\ \vartheta(t+1) = \varphi[u(t) + \theta)] + v(t+1) + w(t+1) - \theta \end{cases} \tag{2.19}$$

for the IF neuron model. The operator L which is defined on $\mathcal{P}(\mathcal{F})$ is called the mean-field propagation operator.

Definition 2.4 *The following equation on* $\mu \in \mathcal{P}(\mathcal{F})$

$$L(\mu) = \mu \tag{2.20}$$

is called the ***mean-field equation (MFE)***

Remark 2.3 *The mean-field equation is the achievement of mean-field approach. To determine the law of an individual trajectory, we suppose that this law governs the interaction of all the units onto the selected one, the resulting law of the selected unit has to be the same law than the generic law. This is summarized in the mean field equation*

$$L(\mu) = \mu$$

Equations 2.18 (resp. (2.19) for the IF model) with the specification of the probability laws define the *mean-field dynamics*. Actually, the law $L(\mu)$ is just the convolution of the probability laws P and the gaussian law g_μ. More precisely, if we apply the discrete time Girsanov theorem 2.20 of the appendix, we have:

Theorem 2.6 $L(\mu)$ *is absolutely continuous with respect to* P *and it density is given by*

$$\frac{dL(\mu)}{dP}(\eta) = \int \exp \frac{1}{\sigma^2} \sum_{t=0}^{T-1} \left[-\frac{1}{2}\xi(t+1)^2 + \Phi_{t+1}(\eta)\xi(t+1) \right] dg_\mu(\xi) \tag{2.21}$$

PROOF : The proof is essentially a simplified version of the application of the finite-time Girsanov theorem which was used to prove theorem (2.4). The conditioning is done here with respect to v which is the difference between the trend terms of the free dynamics and of the mean-field dynamics. ■

Remark 2.4 *We have to notice for further use that*

$$\Gamma(\mu) = \int \log \frac{dL(\mu)}{dP}(\eta) d\mu(\eta) \tag{2.22}$$

In all the cases, for $0 < t < T$ the projection of the laws $\gamma(\mu)$ and $L(\mu)$ on the $t+1$ first instants just depend on the projection of μ on the t first instants. Since the projection of μ on the initial instant is always μ_{init}, the projection of $L(\mu)$ on the two first instants $\{0,1\}$ depend ony on μ_{init} and similarly, the projection of $L^t(\mu)$ on the $t+1$ first instants $\{0,1,...,t\}$ depends only on μ_{init}. Eventually $\mu_T = L^T(\mu) = L^T(P)$ depends only on μ_{init} and it is the only fixed point of the mean-field propagation operator L. So we have shown the following

Theorem 2.7 *The probability measure μ_T=$L^T(P)$ is the only solution of the mean-field equation with initial condition μ_{init}.*

2.3.3 LDP for RRNN mean-field theory

In this section, we fully use the computation results of the previous section to show the rigorous foundations of mean-field theory for RRNN. The approach is the following:

(a) The empirical measure μ_u of the network dynamics satisfies a **large deviation principle (LDP)** under $P^{\otimes N}$ with the good rate function $\mu \in \mathcal{P}(\mathcal{F}) \to I(\mu, P) \in \mathbb{R}^+$. Actually, when the size of the network goes to infinity, the empirical measure converges in law exponentially fast towards P. The definition of LDP and its consequences are outlined in the appendix in definition 2.16. Sanov theorem is stated in appendix, theorem 2.25.

(b) According to corollary 2.5, the density of the new law of μ_u with respect to the original law when we switch from $P^{\otimes N}$ that governs the free dynamics to Q_N that governs the RRNN dynamics is $\exp N\Gamma(\mu)$.

(c) Combining (a) and (b), one obtains that under Q_N, the sequence μ_u satisfies a LDP with the good rate function

$$H(\mu) = I(\mu, P) - \Gamma(\mu) \tag{2.23}$$

This kind of result is used in statistical physics under the name of **Gibbs variational principle** [16]. The functional H is called the **free energy**. Notice, that the classical statistical mechanics framework is relative to equilibrium state. It is applied here for state trajectory. For that reason, this approach is called the **dynamic mean-field theory** [31]. Here, it is be quite technical to support it rigorously. One has to show that H is is lower semi-continuous and is a good rate function (see Varadhan's theorem 2.24 of the appendix). This kind of proof is rather technical (see [5] for a general approach and [27] for the proof for AFRRNN model). So we admit the following result

Theorem 2.8 *Under the respective laws Q_N the family of empirical measures (μ_N) of $\mathcal{P}(\mathcal{F})$ satisfies a full large deviation principle with the good rate function H*

(d) It is clear from remark 2.4 that $H(\mu_T) = 0$ where μ_T is is the unique solution of MFE with initial condition μ_{init}, so it is the fixed point of L. Thus μ_T is a minimum of H.

The basic computation is the following: first we apply the definition 2.19 of the relative entropy that is given in in the appendix

$$I(\mu_T, P) = \int \log \frac{d\mu_T}{dP}(\eta) d\mu_T(\eta)$$

Since μ_T is the solution of MFE, we have

$$\frac{d\mu_T}{dP}(\eta) = \frac{dL(\mu_T)}{dP}(\eta)$$

then we apply the previous remark 2.4 which states

$$\Gamma(\mu_T) = \int \log \frac{dL(\mu_T)}{dP}(\eta) d\mu_T(\eta)$$

to check

$$I(\mu_T, P) = \Gamma(\mu_T) \Rightarrow H(\mu_T) = 0$$

(e) To obtain the exponential convergence of the sequence of empirical measures μ_u under Q_N when $N \to \infty$, one has eventually to show that $H(\mu) = 0 \Rightarrow \mu = \mu_T$. This point is technical too. It is proved in a similar still more general framework (continuous time) in [5] using a Taylor expansion. The same method is and applied to show the unicity for AFRRNN model in [27].

Thus, we have the main result of that section:

Theorem 2.9 *When the size N of the network goes to infinity, the sequence of empirical measures (μ_u) converges in probability exponentially fast towards μ_T which is the unique solution of the mean-field equation $L(\mu) = \mu$*

The practical implications of theorem 2.9 are not straightforward. What about the "local chaos" or the "vanishing correlation" anzats ? We built the mean-field dynamics and obtained the limit μ_T by assuming such an hypothesis to use central limit theorem.

2.3.4 Main results of RRNN mean-field theory

First notice that theorem 2.9 may be extended to RRNN with fast decreasing connection weight distribution. More precisely, let us set

Hypothesis 2.10 (H) *If for all N, the common probability law ν_N of the connexion weights satisfies*

$$\begin{array}{ll}(i) & \int w d\nu_N(w) = \frac{\overline{v}}{N} \\ (ii) & \int w^2 d\nu_N(w) = \frac{v^2}{N} + \frac{\overline{v}^2}{N^2} \\ (iii) & \exists a > 0, \exists D > 0 \text{ such that } \int \exp aNw^2 d\nu_N(w) \leq D\end{array}$$

then the family (ν_N) is said to satisfy hypothesis (H)

Then, it is possible to show (see [26] and [27] that when hypothesis (H) is checked by the AFRRNN model then the exponential convergence theorem 2.9 is still valid. This assumption is useful to extend mean-field theory to diluted RRNN with sparse connections.
From theorem 2.9 two important results may be deduced rigorously. The first one is a "propagation of chaos" result which support the basic intuition of mean field theory about the asymptotic independance of finite subsets of indiviuals when the population size grows to infinity.

Theorem 2.11 *Let k be a positive integer and $(f_i)_{i\in\{1,\ldots,k\}}$ be k continuous bounded functions on $\mathcal{F}$, when the size N of the network goes to infinity, then*

$$\int \prod_{i=1}^{k} f_i(u_i) dQ_N(u) \to \prod_{i=1}^{k} \int f_i(\eta) d\mu_T(\eta)$$

PROOF : The idea of the proof is due to Sznitman [32].
First, a straightforward consequence of theorem 2.9 is that when we apply the sequence of random measures (μ_N) to the test function F on $\mathcal{P}(\mathcal{F})$ defined by $F(\mu) = \prod_{i=1}^{k} \int f_i(u_i) d\mu(u)$ we get the convergence of

$$\lim_{N\to\infty} \int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j) \right] dQ_N(u) = \prod_{i=1}^{k} \int f_i(\eta) d\mu_T(\eta)$$

Thus it remains to compare $\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j)\right] dQ_N(u)$ and $\int \prod_{i=1}^{k} f_i(u_i) dQ_N(u)$

From the symmetry property of Q_N, it is clear that for any subset $\{j_1, ..., j_k\}$ of k neurons among N, we have

$$\int \prod_{i=1}^{k} f_i(u_{j_i}) dQ_N(u) = \int \prod_{i=1}^{k} f_i(u_i) dQ_N(u)$$

If we develop $\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j)\right] dQ_N(u)$, we get

$$\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j)\right] dQ_N(u) = \frac{1}{N^k} \sum_{\{j_1,...,j_k\}} \int \prod_{i=1}^{k} f_i(u_{j_i}) dQ_N(u) \quad (2.24)$$

The average sum in (2.24) is here over all applications of $\{1, ..., k\}$ in $\{1, ..., N\}$. And the equality is proven if we replace it by the average over all injections of $\{1, ..., k\}$ in $\{1, ..., N\}$, since the terms are all equal for injections. But when N goes to infinity the proportion of injections which is $\frac{N!}{(N-k)!N^k}$ goes to 1 and thus the contributions of repetive k-uple is neglectible wwhen ^{n} is large. Therefore

$$\lim_{N\to\infty} \left[\int \prod_{i=1}^{k} \frac{1}{N} \left[\sum_{j=1}^{N} f_i(u_j)\right] dQ_N(u) - \int \prod_{i=1}^{k} f_i(u_i) dQ_N(u) \right] = 0$$

■

Still, this propagation of chaos result is valid when the expectation of the test function is taken with respect to the connection law. Thus, it doesn't say anything precise about the observation relative to a single large-size network.

Actually, since exponentially fast convergence in probability implies almost sure convergence form Borel-Cantelli lemma, we are able to infer the following statement from theorem 2.9. Recall that we note (as in the proof of theorem 2.4) $Q_N(\mathcal{J})$ the conditional law of the network state trajectory given $\mathcal{J}$ the system of synaptic weight and we dfine $\mu_N(u) = \frac{1}{N}\sum_{i=1}^{N} \delta_{u_i}$ for the empirical measure on $\mathcal{F}$ which is associated to a network trajectory $u \in \mathcal{F}^N$.

Theorem 2.12 *Let F be a bounded continuous functionnal on $\mathcal{P}(\mathcal{F})$, we have almost surely in $\mathcal{J}$*

$$\lim_{N\to\infty} \int F[\mu_N(u)] dQ_N(\mathcal{J})(u) = F(\mu_T)$$

Notice we cannot use that theorem to infer a "quenched" propagation of chaos result similar to theorem 2.11 which was an annealed propagation of chaos resuult (i.e. averaged over the connexction weight distribution). It is not possible because for aa given network configuration $\mathcal{J}$, $Q_N(\mathcal{J})$ is no more symmetrical with respect to the individual neurons. Nevertheless, we obtain the following crucial result we apply theorem 2.12 to the case where F is the linear form $F(\mu) = \int f d\mu$

Theorem 2.13 *Let f be a bounded continuous function on $\mathcal{F}$, we have almost surely in $\mathcal{J}$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \int f(u_i) dQ_N(\mathcal{J})(u) = \int f(\eta) d\mu_T(\eta)$$

2.4 Mean-field dynamics for analog networks

Actually, we are interested in the stationary dynamics of large random recurrent neural networks. Moreover since we want to study the meaning of oscillations and of chaos, the regime of low noise is specially interesting since the oscillations are practically cancelled if the noise is too loud. For these reasons, we cannot be pratically satisfied by the obtention of the limit μ_0 of the empirical measures. So we shall extract from μ_0 dynamical informations on the asymptotics of the network trajectories. Notice that the distribution of the connexion weight distribution is not necessarily gasussian as long as it satisfies hypothesis (H:2.10).

2.4.1 Mean-field dynamics of homogeneous networks

General mean-field equations for moments

Recall that in section 2 of that chapter (definition 2.2) we defined for any probability measure $\mu \in \mathcal{P}(\mathcal{F})$ the two first moments of μ, m_μ and c_μ. Let us recall these notations:

$$\begin{cases} m_\mu(t+1) = \overline{v} \int f[\eta(t)] d\mu(\eta) \\ c_\mu(s+1, t+1) = v^2 \int f[\eta(s)] f[\eta(t)] d\mu(\eta) \\ q_\mu(t+1) = c_\mu(t+1, t+1) \end{cases}$$

where f is the sigmoid function $f(x) = \dfrac{e^x}{1+e^x}$

In this section, in order to alleviate notations, we note m, c, q instead of $m_{\mu_0}, c_{\mu_0}, q_{\mu_0}$ where μ_0 is the asymptotic probability that was shown to be

a fixed point of the mean-field evolution operator L in last section. By expressing that μ_0 is a fixed point of L, we shall produce some evolution autonomous dynamics on the moments m, c, q.
More precisely we have from the definition of L (see definition 2.3 in section 3) that the law of $\eta(t)$ under μ_0 is a gaussian law of mean $m(t) - \theta$ and of variance $q(t) + \sigma^2$ (see equations 2.17 and 2.18). So we have

$$\begin{cases} m(t+1) = \overline{v} \int f[\sqrt{q(t)+\sigma^2}\xi + m(t) - \theta] d\gamma(\xi) \\ q(t+1) = v^2 \int f[\sqrt{q(t)+\sigma^2}\xi + m(t) - \theta]^2 d\gamma(\xi) \end{cases} \tag{2.25}$$

where γ is the standard gaussian probability on $\mathbb{R}$: $d\gamma(\xi) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\xi^2}{2}\right] d\xi$
Moreover, the covariance of $(\eta(s), \eta(t))$ under μ_0 is $c(s,t)$ if $s \neq t$. Thus in that case, considering the standard integration formula of a 2d gaussian vector:

$$E[f(X)g(Y)] = \int\int \quad f\left(\sqrt{\frac{Var(X)Var(Y)-Cov(X,Y)^2}{Var(Y)}}\xi_1 + \frac{Cov(X,Y)}{\sqrt{Var(Y)}}\xi_2 + E(X)\right\} ... \\ g[\sqrt{Var(Y)}\xi_2 + E(Y)] d\gamma(\xi_1) d\gamma(\xi_2)$$

we obtain we obtain the following evolution equation for covariance:

$$c(s+1,t+1) = v^2 \int\int \quad f\left(\sqrt{\frac{[q(s)+\sigma^2][q(t)+\sigma^2]-c(s,t)^2}{q(t)+\sigma^2}}\xi_1 + \frac{c(s,t)}{\sqrt{q(t)+\sigma^2}}\xi_2 + m(s) - \theta\right) ... \\ f[\sqrt{q(t)+\sigma^2}\xi_2 + m(t) - \theta] d\gamma(\xi_1) d\gamma(\xi_2) \tag{2.26}$$

The dynamics of the mean-field system (2.25,2.26) can be studied in function of the parameters:

- the mean $\overline{v}$ of the connexion weights,
- the standard deviation v of the connexion weights
- the firing threshold θ of neurons.

Notice that the time and size limits do not necessarily commute. Therefore, any result on long time dysnaics of the mean-field system may not be an exact prediction of the large-size limit of stationary dynamics of random recurrent networks. However, for our model, extensive numerical simulations have shown ([10],[12]) that the time asymptotics of the mean-field system is informative about moderately large random recurrent network stationary dynamics (from size of some hundred neurons).
More precisely, in the low noise limit ($\sigma << 1$), two points of view are interesting:

- the ensemble stationary dynamics is given by the study of the time asymptotics of the dynamical system

$$\begin{cases} m(t+1) = \overline{v} \int f[\sqrt{q(t)}\xi + m(t) - \theta] d\gamma(\xi) \\ q(t+1) = v^2 \int f[\sqrt{q(t)}\xi + m(t) - \theta]^2 d\gamma(\xi) \end{cases} \tag{2.27}$$

- the synchronization of the individual neuron trajectories. Actually, the $m(t)$ and $q(t)$ may converge when $t \to \infty$ towards limits m^* and q^* (stable equilibria of the dynamical system 2.27) with a great variety of dynamical behaviours. Each individual trajectory may converge to a fixed point and (m^*, q^*) are the statistical moments of the fixed point empirical distributions. Another case is provided by individual chaotic oscillations around m^* where q^* measures the amplitude of the oscillations.

The discrimination between these two situations which are very different from the point of view of neuron dynamics is given by the study of the mean quadratic distance which will be outlined in the next paragraph.

Study of the mean quadratic distance

The concept of the mean quadratic distance was introduced by Derrida and Pommeau in [14] to sudy the chaotic dynamics of extremely diluted large size networks. The method originates to check the sensitivity of the dynamical system to initial conditions. The idea is the following: let us consider two networks trajectories $u^{(1)}$ and $u^{(2)}$ of the same network configuration which is given by the synaptic weight matrix (J_{ij}). Their mean quadratic distance is defined by

$$d_{1,2}(t) = \frac{1}{N} \sum_{i=1}^{N} [u_i^{(1)}(t) - u_i^{(2)}(t)]^2$$

For a given configuration, if the network trajectory converges towards a stable equilibrium or towards a limit cycle (synchronous individual trajectories), then the mean quadratic distance between closely initialized trajectories goes to 0 when times goes to infinity. On the contrary, when this distance goes far from 0, for instance converges towards a non zero limit, whatever close the initial conditions are, the network dynamics present in some sense "sensitivity to initial conditions" and thus this behaviour of the mean quadratic distance can be considered to be symptomatic of chaos. We apply this idea in [9] to characterize instability of random recurrent neural network.

In the context of large deviation based mean-field theory, the trajectories $u^{(1)}$ and $u^{(2)}$ are submitted to independant synaptic noises and the mean quadratic distance is defined by

$$d_{1,2}(t) = \frac{1}{N}\sum_{i=1}^{N}\int [u_i^{(1)}(t) - u_i^{(2)}(t)]^2 dQ_N^{(1,2)}(u^{(1)}, u^{(2)}) \tag{2.28}$$

where $Q_N^{(1,2)}$ is the joint probability law on $\mathcal{F}^{2N}$ of the network trajectories $(u^{(1)}, u^{(2)})$ over the time interval $\{0, ..., T\}$. Following the same lines as in last sections, it is easy to show a large deviation principle for the empirical measure of the sample $(u_i^{(1)}, u_i^{(2)})_{i\in\{1,...,N}$ under $Q_N^{(1,2)}$ when $N \to \infty$. Then we get the almost sure convergence theorem

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\int f_1(u_i^1) f_2(u_i^2) dQ_N(\mathcal{J})(u) = \int f_1(\eta_1) f_2(\eta_2) d\mu_T^{(1,2)}(\eta_1, \eta_2)$$

where $\mu_T^{(1,2)}$ is the fixed point of the mean-field evolution operator $L^{(1,2)}$ of the joint trajectories which is defined on the probability measure set $\mathcal{P}(\mathcal{F}\times\mathcal{F})$ exactly in the same way as L was defined previously in definition 2.3.
Then if we define the instantaneous covariance between two trajectories by

Definition 2.5 *The instantaneous cross covariance between the two trajectories under their joint probability law is defined by*

$$c_{1,2}(t) = \int \eta_1(t)\eta_2(t) d\mu_T^{(1,2)}(\eta_1, \eta_2) \tag{2.29}$$

where $\mu_T^{(1,2)}$ is the fixed point measure of the joint evolution operator $L^{(1,2)}$ defined from an initial condition $\mu_{init}^{(1,2)}$.

Then we can follow the argument, which was already used for the covariance evolution equation (2.26). Thus we obtain the following evolution equation for the instantaeous cross covariance equation

$$\begin{aligned} c_{1,2}(t+1) = \upsilon^2 \int\int \quad & f\left(\sqrt{\tfrac{[q_1(t)+\sigma^2][q_2(t)+\sigma^2]-c_{1,2}(t)^2}{q_2(t)+\sigma^2}}\xi_1 + \tfrac{c_{1,2}(t)}{\sqrt{q_2(t)}}\xi_2 + m_1(t) - \theta\right) ... \\ & f[\sqrt{q(t)+\sigma^2}\xi_2 + m_2(t) - \theta] d\gamma(\xi_1) d\gamma(\xi_2) \end{aligned} \tag{2.30}$$

The proof is detailed in [26].
It is obvious now to infer the evolution of the mean quadratic distance from the following square expansion

Proposition 2.14 *The mean quadratic distance obeys the relation*

$$d_{1,2}(t) = q_1(t) + q_2(t) - 2c_{1,2}(t) + [m_1(t) - m_2(t)]^2$$

Study of the special case of balanced inhibition

In order to show how the previous equations are used we shall display the special case of balanced inhibition and excitation. The study of the discrete time 1-dimensional dynamical system with different parameters was adressed in the previous chapter. See also ([10] and [12]) for more details.
We choose in the previous model the special case where $\overline{v} = 0$. This choice simplifies considerably the evolution study since $\forall t, m(t) = 0$ and the recurrence over $q(t)$ is autonomous. So we have just to study the attractors of a single real function.
Moreover, the interepretation of $\overline{v} = 0$ is that there is a general balance in the network between inhibitory and excitatory connections. Of course, the model is still far from biological plausibility since the generic neuron is endowed both with excitatory and inhibitory functions. In next section, the model with several populations will be adressed. Nevertheless, the case $\overline{v} = 0$ is of special interest. In the limit of low noise, the system amount to the recurrence equation:

$$q(t+1) = v^2 \int f[\sqrt{q(t)}\xi - \theta]^2 d\gamma(\xi) \tag{2.31}$$

we can scale $q(t)$ to v and we obtain

$$q(t+1) = \int f[v\sqrt{q(t)}\xi - \theta]^2 d\gamma(\xi) = h_{v,\theta}[q(t)] \tag{2.32}$$

where the function $h_{v,\theta}$ of $\mathbb{R}^+$ into $\mathbb{R}^+$ is defined by

$$h_{v,\theta}(q) = \int f[v\sqrt{q(t)}\xi - \theta]^2 d\gamma(\xi)$$

This function is positive, increasing and goes to 0.5 when q goes to infinity. The recurrence (2.32) admits on $\mathbb{R}^+$ a single stable fixed point $q^*(v, \theta)$. This fixed point is increasing with v and decreasing with θ. We represent in figure 2.1 the diagram of the variations of function $q^*(v, \theta)$. It is obtained from a numerical simulation with a computation of $h_{v,\theta}$ by Monte-Carlo method.
Let us now consider the stablity of the network dynamics by studying the covariance and the mean quadratic distance evolution equation. The covariance evolution equation (2.26) in the low noise limit and when $t \to \infty$

Figure 2.1: Variations of the fixed point $q^*(\upsilon, \theta)$ in fonction of the network configuration parameters

amounts to

$$c(s+1,t+1) = \upsilon^2 \int\int \begin{array}{l} f\left(\sqrt{\frac{q^{*2}-c(s,t)^2}{q^*}}\xi_1 + \frac{c(s,t)}{\sqrt{q^*}}\xi_2 - \theta\right) ... \\ f\left(\sqrt{q^*}\xi_2 - \theta\right) d\gamma(\xi_1) d\gamma(\xi_2) \end{array} \tag{2.33}$$

Let us scale the covariance with υ^2 we obtain the recurrence

$$c(s+1,t+1) = H_{\upsilon,\theta,q}(c(s,t))$$

with

$$H_{\upsilon,\theta,q}(c) = \int\int f\left(\upsilon\sqrt{\frac{q^2-c^2}{q}}\xi_1 + \frac{c}{\sqrt{q}}\xi_2 - \theta\right) f\left(\upsilon\sqrt{q}\xi_2 - \theta\right) d\gamma(\xi_1) d\gamma(\xi_2) \tag{2.34}$$

It is clear from comparing with equation (2.31) that q^* is a fixed point of $H_{\upsilon,\theta,q}$. To study the stability of this fixed point, standard computation shows that

$$\frac{dH_{\upsilon,\theta,q^*}}{dc}(q^*) = \int f'\left(\upsilon\sqrt{q^*}\xi_2 - \theta\right)^2 d\gamma(\xi) \tag{2.35}$$

Then as it is stated in previous chapter, the condition $\frac{dH_{\upsilon,\theta,q^*}}{dc}(q^*) \leq 1$ is a necessary and sufficient condition for the stability of q^*. A detailed and rigorous proof for $\theta = 0$ is provided in [26]. Then two cases occur.

- In the first case where $\frac{dH_{\upsilon,\theta,q^*}}{dc}(q^*) \leq 1$,the stationary limit of $c(s+\tau, t+\tau)$ when $\tau \to \infty$ does not depend on $t-s$ and is $c^* = q^*$. The stationary limit of the mean-field gaussian process is a random point. Its variance is increasing with υ and decreasing with .

- In the second case where $\frac{dH_{\upsilon,\theta,q^*}}{dc}(q^*) > 1$ does not depend on $t-s$ when $t-s \neq 0$ and is equal to $c^* < q^*$. The stationary limit of the gaussian process is the sum of a random point and of a white noise. From the dynamical system point of view, this corresponds to chaotic regime. The signature of chaos is given by the evolution of the mean quadratic distance. The instantaneous covariance converges also towards c^*. Therefore the mean quadratic distance converges towards a non null limit, which is independant of the initial condition distance.

The figures 2.1 and 2.2 shows the evolution of q^* and $q^* - c^*$ in function of υ and θ. When υ is small, there is no bifurcation to chaos. When υ

Figure 2.2: Variations of $q^* - c^*$ in fonction of the network configuration parameters υ and θ

is larger, the bifurcation toward chaos occurs when θ is decreasing. When υ is growing, the bifurcation toward chaos occurs for increasing θ values. Figure **??** of previous chapter shows the interest of variation of input (which is equivalent to threshold variation) allows to hold up the bifurcation to chaos.

2.4.2 Mean-field dynamics of 2-population AFRRNN

2-population AFRRNN model

As it was announced previously, the assumption of a homogeneous connexion weight model is not plausible. Besides in litterature, RRNN models with several neuron populations have been studied as early as in 1977 with [2] aand have been thoroughly investigated in the last decade (see for instance [20]). The heterogeneity of neuron population induce interesting and complex dynamical phenomena such as synchronization.
Actually the mean-field theory that was developped herebefore in the previous sections may be extended without major difficulty to several neuron populations. To give a practical idea of what can be obtained such extensions we consider here two populations with respectively $N_1 = \lambda N$ and $N_2 = (1-\lambda)N$ neurons where $\lambda \in]0,1[$ and where $N \to \infty$.
Four connexion random matrixes have to be considered in this model $\mathcal{J}_{11}, \mathcal{J}_{12}$, $\mathcal{J}_{21}, \mathcal{J}_{22}$ where $\mathcal{J}_{ij}$ is the matrix of connexion weights from population j neuron to population i neuron. The random matrix $\mathcal{J}_{ij}$ is a $(N_j \times N_i$ random matrix with independant indentically distributed entries. Their distribution is governed by statistical parameters $(\overline{\upsilon}_{ij}, \upsilon_{ij})$ and obeys hypothesis (2.10). they are independant altogether.
Yet, technical hypothesis (H) does nnot allow to embed connexion weight a rigorously constant sign to distinguish between inhibitory and excitatory neurons. Actually there is no probability distribution on positive real numbers with mean and variance respectively scaled as $\frac{\overline{\upsilon}}{N}$ and $\frac{\upsilon^2}{N}$. Thus, the positivity of the support induces on the other side of the distribution a heavy tail which will not respect assumption (iii) in hypothesis (H). However, it is possible to consider probability distributions which are checking hypothesis (H) and which are loading the negative numers (or alternatively) the positive ones) with arbitrary small probability.
We consider here a 2-population model with a population of excitatory neurons and a population of inhibitory neurons (up to the above restriction).

General mean-field equations for moments

A large deviation principle may be obtained for the 2-population model for gaussian connexion weights. So the convergence in finite time to the mean-field dynamics is shown in the model that is described in the previous paragraph according to the same proof as in the previous 1-population model. See [26] for a rigorous proof and [11] for a more practical statement of results. The limit of the empirical measure is the law of a gaussian vector which takes its values in $\mathcal{F} \times \mathcal{F}$. Each factor stands to describe the repartition of a neural population. Note that the two components are independant. As for the 1-population model we note $m_k(t), q_k(t), c_k((s,t)$ the mean, variance and covariance at given times of the empirical measure of population k ($k \in \{1,2\}$). The mean-field evolution equation for these moments is described by the following system:

$$\left\{\begin{array}{ll} m_k(t+1) = & \sum_{j\in\{1,2\}} \overline{v}_{kj} \int f[\sqrt{q_j(t)+\sigma^2}\xi + m_j(t) - \theta_j] d\gamma(\xi) \\ q_k(t+1) = & \sum_{j\in\{1,2\}} v_{kj}^2 \int f[\sqrt{q_j(t)+\sigma^2}\xi + m_j(t) - \theta_j]^2 d\gamma(\xi) \\ c_k(s+1,t+1) = & \sum_{j\in\{1,2\}} v_{kj}^2 \int\int f\left(\sqrt{\frac{[q_k(s)+\sigma^2][q_k(t)+\sigma^2]-c_k(s,t)^2}{q_k(t)+\sigma^2}}\xi_1 \ldots\right. \\ & \left. + \frac{c_k(s,t)}{\sqrt{q_k(t)+\sigma^2}}\xi_2 + m_k(s) - \theta_k\right) \ldots \\ & f[\sqrt{q_k(t)+\sigma^2}\xi_2 + m_k(t) - \theta_k] d\gamma(\xi_1) d\gamma(\xi_2) \end{array}\right. \tag{2.36}$$

Results and discussion

As far as numerical studies are concerned, we choose the following values for their statistical moments

$$\left\{\begin{array}{llll} \overline{v}_{1,1} = & gd & v_{1,1} = & g \\ \overline{v}_{1,2} = & -2gd & v_{1,2} = & \sqrt{2}g \\ \overline{v}_{2,1} = & gd & v_{2,1} = & g \\ \overline{v}_{22} = & 0 & v_{22} = & 0 \end{array}\right. \tag{2.37}$$

In this study, according to some biological scheme, excitatory neurons are connected both to excitaztory neurons and inhibitory neurons and inhibitory neurons are both connected to excitatory neurons. Moreover, the number of parameters is reduced to allow numerical exploration of the synchornization parameter. We keep two independant parameters:

- g stands for the non linearity of the transfer function
- d stands for the differentiaton of the two populations (inhibitory vs. excitatory).

Considering the firing thresholds as previously, there is no variation about individual thesholds. Excitatory neuron threshold θ_1 is chosen equal to 0 and inhibitory neuron threshold θ_2 is chosen equal to 0.3 because thge activation potential of inhibitory neurons is always positive.
In the bifurcation map of 2.3 (extracted from 01 several dynamical regimes are displayed and the corresponding numerical ranges of paarameters d nd g are displayed. Notice that theoretical previsions of the mean-field equations (2.36) and the large scale simulations of large-size network behaviour are consistent. The occurence of fixed point and chaos with a fixed point to chaos bifurcation (with a narrow transition route) is confirmed for weak d (in accordance with homogeneous network study). When differentiation paramerter d is sufficient (about 2), fixed point looses its stability through a Hopf bifurcation to give rise to synchronous oscillations when g is growing. Moreover, a new phenomenon is displayed thank to the RRNN modelization. For large g, there is a significant transition regime between stationary chaos and synchronised oscillations which is named "*cyclostationary chaos*". In that regime statistical parameters are exhibitting regular periodic oscillations though individual trajectories are diverging with a mean quadratic distance behaviour which is characteristic from chaos.

Figure 2.3: Bifurcation map of the 2-population model

2.5 MFT-based oscillation analysis in IF networks.

In this section we would like to give an interesting application of mean-field approaches for spiking neurons. It was developped in [7]. This paper is part of a current of research which studies the occurence of synchronized oscillations in recurrent spiking neural networks [4, 3, 6] in order to give an account of spatio-temporal synchronization effects, which are observed in many situations in neural systems [18, 29, 8, 28].

2.5.1 IFRRNN continuous-time model

The model of [7] is in continuous time. There is no synaptic noise but the neurons are submitted to a random external output. So, equation (2.9)

$$u(t+1) = \varphi[u(t) + \theta)] + v(t+1) + w(t+1) - \theta \tag{2.38}$$

where

- $\gamma \in]0, 1[$ is the **leak**
- φ is defined by $\varphi(u) = \begin{cases} \gamma u \text{ if } \frac{\vartheta}{\gamma} < u < \theta \\ \vartheta \text{ else} \end{cases}$
- ϑ is the reset potential and $\vartheta < 0 < \theta$

has to be replaced by

$$\begin{cases} u(t) < \theta & \Rightarrow \tau \dot{u}(t) = -u(t) + v_{net}(t) + v_{ext}(t) \\ u(t-0) = \theta & \Rightarrow u(t+0) = \vartheta \end{cases} \tag{2.39}$$

where

- τ is the **characteristic time** of the neuron
- v_{net} is the synaptic input from the network
- v_{ext} is the external input
- ϑ is the reset potential and $0 < \vartheta < \theta$. Note that $u(t-0)$ and $u(t+0)$ are respectively the left and right limits of u at firing time t. Thus, the refractory period is assumed to be zero.

This model of continuous time neuron dynamics is introduced in chapter 1, section (**??**).

Moreover, since the inputs are modelled by continuous-time stochastic processes, equation (2.39) is a stochastic differential equation of the type

$$\tau du(t) = -u(t)dt + dV_t \tag{2.40}$$

with $dV(t) = dV_{ext}(t) + dV_{net}(t)$
Now we shall explicit these stochastic processes in order to obtain the Fokker-Planck equation of the network dynamics in mean-field approximation.

2.5.2 Modelling the external input

The network is a recurrent inhibitory network and we study its reaction to random excitatory synaptic inputs. We suppose that in the network each neuron receives excitations from C_{ext} external neurons connected via constant excitatory synapses J_{ext}. The corresponding external current is a Poisson process with emission frequency ν_{ext}.
Let us examine the effect of a superposition of a large number C of independant identically distributed low-rate ν Poisson processes. Put

$$\mathcal{I}(t) = J \sum_{i=1}^{C} \mathcal{N}_i(t)$$

where $\mathcal{N}_i(t)$ are i.i.d. Poisson processes with firing rate ν.
Then $I(t)$ is a stochastic process with independant stationary increments such that $\mathrm{E}(\mathcal{I}(t)) = \mu t = JC\nu t$ and $\mathrm{Var}(\mathcal{I}(t)) = \sigma^2 t = J^2 C\nu t$.
Thus $\mu = JC\nu$ and $\sigma = J\sqrt{C\nu}$.
We are interested in studying such processes when they reach the firing threshold θ which is far greater than the elementary increment J. In typical neural applications, $J = 0.1$ mv and $\theta = 20$ mV. At this level, operating a classical time-space rescaling, $\mathcal{I}(t)$ appears like a gaussian process with independant increments and same moments. We have

$$d\mathcal{I}(t) \sim \mu dt + \sigma dB_t$$

where (B_t) is the standard brownian motion. If we apply the subsequent to the external synaptic input we get the following modelling in the limit of large size and low rate

$$dV_{ext}(t) = \mu_{ext} dt + \sigma_{ext} dB(t)$$

with $\mu_{ext} = J_{ext} C_{ext} \nu_{ext}$ and $\sigma_{ext} = J_{ext}\sqrt{C_{ext}\nu_{ext}}$.

2.5.3 Mean-field approximation of the internal input

In the framework of continuous-time modelling, the synaptic input definition of v_{net} for IF neuron i which was according to equation (2.3)

$$v_i(\mathcal{J}, u)(t+1) = \sum_{j=1}^{N} J_{ij} x_j(t)$$

has to be replaced by

$$v_i(\mathcal{J}, u)(t) = \tau \sum_{j=1}^{N} J_{ij} \sum_k \delta\left(t - T_j^k(u) - D\right) \tag{2.41}$$

where

- δ is the dirac distribution,
- $T_j^k(u)$ are the successive firing times of neuron j during the networrk trajectory u,
- D is the synaptic transmission delay.

Mean-field approximation in the finite time set framework consisted in previous sections in finding a fixed point for the mean-field propagation operator L, namely in

- approximating random vectors v_i by gaussian vectors of law g_μ where μ is a probability law on the individual neuron potential trajectory space (finite-dimensional vector space)
- finding μ as the probability law of the neuron dynamical equation with this approximation for the synaptic input

The synapses between neurons are all negative (inhibitory), with the same value $-J < 0$. They are sparse. Each neurons receives $C << N$ connections, where C is a fixed integer, and N is the total number of neurons. I_i is the synaptic current coming from the other neurons.
The mean-field approximation in [7] follows the same logic. The network is supposed to be sparsely connected. All the connexion weights are equal to $-J$ as soon as they are non null. Each neuron is connected to C neurons which are randomly drawn among the network with $C << N$ connections, where C is a fixed integer and N is the total number of neurons. Another model is considered further where the connection weights are independant

random variables equal to $-J$ with probability $\frac{C}{N}$ and to 0 else. We shall focus here on the first model.

The first step of the mean field approximation consists for a given rate function ν in defining the non stationary gaussian process

$$dV_{net}(t) = \mu_{net}(t)dt + \sigma_{net}(t)dB(t) \tag{2.42}$$

where

- the drift μ_{net} is given by

$$\mu_{net}(t) = -CJ\nu(t-D)\tau \tag{2.43}$$

- and where the diffusion coefficient σ_{net} is given by

$$\sigma_{net}(t)^2 = J^2C\nu(t-D)\tau \tag{2.44}$$

The second step consists in considering the following diffusion with "tunnelling effect"

$$\begin{cases} u(t) < \theta & \Rightarrow \tau du(t) = -u(t)dt + dV_{net}(t) + dV_{ext}(t) \\ u(t-0) = \theta & \Rightarrow u(t+0) = \vartheta \end{cases} \tag{2.45}$$

From the solution of this equation, a rate function ν of the probability rate of absorption by the barrier θ at time t is inferred.

2.5.4 Fokker-Planck equation

Closed form equation

Note $p(u,t)$ the probability density of the solution $u(t)$ of (2.45). Define

$$\begin{aligned} \mu(t) &= \mu_{net}(t) + \mu_{ext}(t) \\ \sigma(t) &= \sqrt{\sigma_{net}(t)^2 + \sigma_{ext}(t)^2} \end{aligned}$$

Then $p(u,t)$ is solution of the Fokker-Planck equation for diffusion process for $u < \theta$ and $u \neq \vartheta$:

$$\frac{\partial p}{\partial t}(u,t) = \frac{\sigma(t)^2}{2}\frac{\partial^2 p}{\partial u^2}(u,t) + \frac{\partial}{\partial u}\left[(u-\mu(t))p(u,t)\right] \tag{2.46}$$

The tunelling effect from θ to ϑ is taken into acccount in the following boundary conditions

$$\begin{cases} p(\theta,t)=0 \\ \frac{\partial p}{\partial u}(\vartheta+0,t)=\frac{\partial p}{\partial u}(\vartheta-0,t)+\frac{\partial p}{\partial u}(\theta-0,t) \end{cases} \tag{2.47}$$

Last the firing rate is defined by

$$\nu(t)=\frac{\partial p}{\partial u}(\theta-0,t) \tag{2.48}$$

Stationary solution

It is easy to find the stationary solution of the previous equation

$$\frac{\partial p}{\partial t}(u,t)=0$$

Suppose a given constant firing rate ν_0, then set

$$\begin{cases} \mu_0=-CJ\nu_0\tau+\mu_{ext} \\ \sigma_0=\sqrt{CJ^2\nu_0\tau+\sigma_{ext}^2} \end{cases} \tag{2.49}$$

and plug it into the differential second order equation

$$\frac{\sigma_0^2}{2}\frac{d^2p}{du^2}+\frac{d}{du}\left[(u-\mu_0)p(u)\right]=0 \tag{2.50}$$

with the following boundary conditions

$$\begin{cases} p(\theta)=0 \\ \frac{dp}{du}(\vartheta+0)=\frac{dp}{du}(\vartheta-0)+\frac{dp}{du}(\theta-0,t) \end{cases} \tag{2.51}$$

One obtains easily the following stationary distribution

$$\begin{cases} For\ u<\vartheta, \quad p(u)=\frac{2\nu_0}{\tau}e^{-y_u^2}\int_{\upsilon_\vartheta}^{\upsilon_\theta}e^{y^2}dy \\ For\ u\geq\vartheta, \quad p(u)=\frac{2\nu_0}{\tau}e^{-y_u^2}\int_{y}^{\upsilon_\theta}{}_{u}e^{y^2}dy \end{cases}$$

where $y_u=\frac{u-\mu_0}{\sigma_0}, y_\vartheta=\frac{\vartheta-\mu_0}{\sigma_0}$ and $y_\theta=\frac{\theta-\mu_0}{\sigma_0}$
Then the normalization condition $\int_{-\infty}^{\infty}p(u)du=1$ allows to infer

$$\frac{1}{\nu_0\tau}=\int_0^{+\infty}e^{-y^2}\left[\frac{e^{2y_\theta y}-e^{2y_\vartheta y}}{y}\right]dy \tag{2.52}$$

The relations (2.49,2.52) allows to compute numerically ν_0. The equation (2.52) can be approximately solved in the situation where the fluctuations

σ_0 are weak (i.e. $y_\theta >> 1$ which means that the spiking events are rare). In this case :

$$\nu_0 \tau \approx \frac{y_\theta}{\sqrt{\pi}} e^{-y_\theta^2} \tag{2.53}$$

This asymptotic expression can be compared to the escape probability from the equation of motion of a particule in a parabolic potential well $\mathcal{V}$, with minimum μ_0, submitted to a brownian excitation

$$\tau dV_t = -(V - \mu_0)dt + \sigma_0 dB_t$$

The time rate to reach $V = \theta$ is thus given by the Arrhenius time

$$\nu_0 \tau \sim e^{-y_\theta^2}$$

Numerical values of ν_0 which are inferred from equations (2.52) and (2.53) are compared in [7] to the result of numerical simulations of the network and there is a good agreement between theoretical predictions and simulated firing rates.

Stability analysis.

The stability analysis for the stationary solution uses normal form technics similar to those described in Chapter 1, but in an infinite dimensional space. The Fokker-Planck is rescaled and expanded around the steady-state solution. This intricate computation is fully detailed in [7] . We simply focus to the results.
The authors find that there is a bifurcation of Hopf type for the stationary solution. Thus, for a certain parameter range, the system exhibits synchronized oscillations of the neurons. A sketch of the bifurcation map is given in figure 2.4 when varying the parameters μ_{ext}, σ_{ext} controling the external excitation.
One can see from that bifurcation diagram that the bifurcation occurs when the drift of the external input is increasing. On the opposite, an increase of the dispersion of the external input stabilizes the steady state. If the external input consists in the superposition of i.i.d. Poisson processes as it was detailed above, then the increase of their common frequency $\nu_e xt$ induces the occurence of an oscillatory regime. There is still a good agreement between the predictions of mean-field theory and the results of simulations.

Figure 2.4: Sketch of the bifurcation diagram of the model (**??**,2.41) when varying the parameters μ_{ext}, σ_{ext} controling the Poisson process of external excitation. SS means Stationary State, while OS means Oscillatory State. The solid line represents the instability line for $D = 0.1\tau$. (Drawn by hand from [7])

2.5.5 Conclusion

Thus, the conclusion is that in this model of a neural network with a sparsely connected inhibitory integrate-and-fire neurons, submitted to a external excitatory Poisson process, and emitting spikes irregularly at a low rate, there is, in the thermodynamic limit, a sharp transition between a regime where the average global is constant, to a synchronized state where neurons are weakly synchronized. The activity becomes oscillatory when the inhibitory feedback is strong enough.
Note that the period of the global oscillations depends on the synaptic transmission delay which cannot be neglected.
Finally, let us mention that the authors performed a finite size analysis of the model and found that global oscillations of *finite coherence time* generically exist *above and below* the critical inhibition threshold.

2.6 Appendix about probability theory

This chapter uses intensively some classical notations and concepts of probability theory. The proofs are omitted but sometimes the results follow from advanced results of this theory. It is not possible to recall here the necessary prerequisites. There are excellent books about probability theory for physicists and engineers such as [23]. We just want here to recall some notations and some results from convergence theory. We have detailed the proof of the "finite-time Girsanov theorem" since it is a crucial result for the chapter.

2.6.1 Elementary Notations

The classical and shortest point of view for considering random phenomena from the 19th century is to consider a random variable x in a space $\mathcal{E}$ as probability law on that space from which all the moments of the law can be computed by integrating the quantity of interest over the probability law of the random variable. For instance, if μ is the probability law of the real random variable x, one has

$$\mathrm{E}(x) = \int_x x d\mu(x)$$

$$\mathrm{E}(x^2) = \int_x x^2 d\mu(x)$$

and more generally for any bounded continuous function f of x

$$\mathrm{E}[\phi(x)] = \int_x \phi(x) d\mu(x)$$

where E is the **mathematical expectation** operator. The expectation of any random variable which takes its value in a topological vector space $\mathcal{F}$ with additional mathematical hypothesis is a vector of $\mathcal{F}$ and the mathematical expectation operator is linear.
More over, if we consider a random vector x taking its value in a finite-dimensional vector space $\mathbb{R}^d$ with probabbility law μ on $\mathbb{R}^d$ we consider its expectation $E(x) \in \mathbb{R}^d$ which is defined by

$$\forall i \in \{1, ..., n\}, [\mathrm{E}(x)]_i = \mathrm{E}(x_i) = \int_{\mathbb{R}^d} x_i d\mu(x)$$

and the symetric $(d, d)-$covariance matrix which is defined by

$$\mathrm{Cov}(x)_{ij} = \mathrm{E}(x_i x_j) - \mathrm{E}(x_i)\mathrm{E}(x_j)$$

Actually, this point of view cannot be used when we are obliged to consider an infinite set of random variables or when we want to operate a variable change. Hence, we are obliged to adopt a more general point of view which was initiated by Kolmogorov in 1933. This approach relies basically upon the consideration of a very large state space Ω which describes all the possible outcomes or states of the world. Then a rich family $\mathcal{A}$ of subsets of Ω is defined such that all the random events of interest are belonging to $\mathcal{A}$. Eventually a probability measure is defined on $\mathcal{A}$ which associates to any random event $A \in \mathcal{A}$ its probability $P(A)$. The triple $(\Omega, \mathcal{A}, P)$ is called a **probability space**.
Later on, we shall have to work on infinite-dimensional space. So let us fix a general framework

Definition 2.6 *A Polish space $\mathcal{F}$ is a metric complete (every Cauchy sequence converges) and separable (there is a denombrable dense subset) space. The $\sigma-$algebra $\mathcal{B}$ of Borel subsets of A Polish space $\mathcal{F}$ is the smallest $\sigma-$algebra that contains the open sets. Given a probability measure μ on the Borel subsets of $\mathcal{F}$ it is possible to integrate any bounded continuous function ϕ on $\mathcal{F}$ and the integral is noted $\int_{\mathcal{F}} \phi(\xi) d\mu(\xi)$. The integral may be extended to a wider class of functions. These functions are called **integrable** with respect to μ.*

.
In that new framework let us define random variables in $\mathcal{F}$.

Definition 2.7 *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathcal{F}, \mathcal{B})$ a Polish space endowed with its Borel $\sigma-$algebra. A **random variable** $x \in \mathcal{F}$ is a state function from Ω into $\mathcal{F}$ such that for any open set B in $\mathcal{F}$, the subset of Ω defined by*

$$(x \in B) = \{\omega \in \Omega \text{ such that } x(\omega) \in \mathcal{B}\}$$

belongs to $\mathcal{A}$ so its probability $P(x \in B)$ is well defined.
*The **probability law** of a random variable $x \in \mathcal{F}$ is the probability law on $\mathcal{F}$ which associates to any Borel subset $B \subset \mathcal{F}$ the probability $P(x \in B)$.*

The probability law of x is noted $x.P$ or P_x. This definition stands for also for general measure than probability laws such as volume measures. More generally, we have

Definition 2.8 *Let $(\Omega, \mathcal{A}, P)$ be a measure space and x amapping from Ω to $\mathcal{F}$ such that*

$$\forall B \in \mathcal{B}, (x \in B) = \{\omega \in \Omega \text{ such that } x(\omega) \in \mathcal{B}\} \in \mathcal{A}$$

Then we define a measure on $(\mathcal{F}, \mathcal{B})$ *that is noted* $x.P$ *or* P_x *by*

$$B \in \mathcal{B} \rightarrow x.P(B) = P_x(B) = P(x \in B)$$

This measure is called the ***image*** *of the measure* P *by the mapping* x

This definition is completed by the following transfer theorem which shows that the mathe-matical expectation can be computed on the state space Ω or on the value space $\mathcal{F}$.

Theorem 2.15 *For any function* ϕ *defined on* $\mathcal{F}$ *and integrable for the probability law* P_x *we have*

$$E[\phi(x)] = \int_\Omega \phi[x(\omega)]dP(\omega) = \int_\mathcal{F} \phi(\xi)dP_x(\xi)$$

The transfer theorem is very useful in theory and in practice. It allows to define the mathematical expectaton of a random variable without any ambiguity.
Kolmogorov's framework allows to define independent random variables by the equivalent following properties

Definition 2.9 *For* $i \in \{1, ..., n\}$ *let* $x_i \in \mathcal{F}_i$ *be random variables, they are said* ***independent*** *if the law* P_x *of the random variable* $x = (x_1, ..., x_n) \in \mathcal{F}_1 \times ... \times \mathcal{F}_n$ *is the product of the* P_{x_i} *which is expressed in the following equivalent properties*

$$P(x \in B_1 \times ... \times B_n) = P_{x_1}(B_1)...P_{x_n}(B_n)$$

$$E[\phi_1(x_1)...\phi_n(x_n)] = E[\phi_1(x_1)]...E[\phi_n(x_n)]$$

2.6.2 Density and Gaussian random vectors

Definition 2.10 *Let* $(\Omega, \mathcal{A}, m)$ *a measure space and* h *an integrable positive function on* Ω *such that* $\int_\Omega h(\omega)dm(\omega) = 1$. *Then we can define a probability measure* Q *on* $(\Omega, \mathcal{A})$ *by*

$$Q(A) = \int_\Omega 1_A(\omega)h(\omega)dm(\omega)$$

Q *is said* ***absolutely continuous*** *with respect to* m, h *is called the* ***density*** *of* Q *with respect to* m *and we can compute the integral for* Q *by using the formula*

$$\int_\Omega \phi(\omega)dQ(\omega) = \int_\Omega \phi(\omega)h(\omega)dm(\omega)$$

We write $\frac{dQ}{dm}(\omega) = h(\omega)$ *or* $dQ((\omega) = h(\omega)dm(\omega)$

Of course, the density functions are commonly used in elementary probability. An important class of probability measures is the Gaussian probability family.

Definition 2.11 *Let $a \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}^+$. The Gaussian probability measure $\gamma = \mathcal{N}(a, \sigma^2)$ is defined by its density with repect to the Lebesgue measure λ on $\mathbb{R}$, which is*

$$\frac{d\gamma}{d\lambda}(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\xi - m)^2}{2\sigma^2}\right]$$

Similarly, this defintion can be extended to d-dimensional vector space and enven to infinite-dimensional Hilbert space. Here, we just need the following

Definition 2.12 *Let $\overline{\theta} \in \mathbb{R}^d$ and K be a $d \times d$ symetric positive matrix, then there exists one and one only probability measure on $\mathbb{R}^d$, which is called the **Gaussian probability** $\gamma = \mathcal{N}(\overline{\theta}, K)$ such that if γ is the probability law of the random vector $x \in \mathbb{R}^n$ then $\forall u \in \mathbb{R}^d$, the law of the random variable $u^t x$[4] is $\mathcal{N}(u^t\overline{\theta}, u^t K u)$.*

Proposition 2.16 *Let x be a random vector with regular gaussian probability $\gamma = \mathcal{N}(\overline{\theta}, K)$ then we have*

$$\begin{cases} E(x) = \int \xi d\gamma(\xi) = \overline{\theta} \\ Cov(x) = E(xx^t) - E(x)E(x)^t = K \end{cases}$$

So a Gaussian law is completely determined by its expectation and its covariance matrix.

Definition 2.13 *With the previous notations, if K is invertible, γ is said to be **regular** and the density of γ with respect to the Lebesgue measure λ is*

$$\frac{d\gamma}{d\lambda}(\xi) = \frac{1}{\sqrt{(2\pi)^n Det(K)}} \exp\left[-\frac{(\xi - m)^t K^{-1}(\xi - m)}{2}\right] \tag{2.54}$$

A common property of the gaussian family is its stability by linear transforms and translation. More precisely, we have

[4] u^t is the transpose of column vectorv u, so $u^t x$ is the scalar product of vectors u and x

Proposition 2.17 *Let x a gaussian random vector which takes its value in the vector space E and Λ a linear mapping of E into F. Then $y = \Lambda x$ is a gaussian random vector in F and*

$$\begin{cases} E(y) = \Lambda E(x) \\ Cov(y) = \Lambda Cov(x)\Lambda^t \end{cases} \tag{2.55}$$

Proposition 2.18 *Let x a gaussian random vector which takes its value in the vector space E and $a \in E$. Then $y = x + a$ is a gaussian random vector in F and*

$$\begin{cases} E(y) = E(x) + a \\ Cov(y) = Cov(x) \end{cases} \tag{2.56}$$

Corollary 2.19 *Let x be a random vector with regular gaussian probability $\gamma = \mathcal{N}(\overline{\theta}, K)$ and let $a \in \mathbb{R}^d$, then the law γ_a of $x+a$ is the regular Gaussian law $\mathcal{N}(\overline{\theta}+a, K)$ and its density with respect to γ can be explicited as follows*

$$\frac{d\gamma_a}{d\gamma}(\xi) = \exp\left[a^t K^{-1}(\xi - \overline{\theta}) - \frac{1}{2}a^t K^{-1} a\right] \tag{2.57}$$

PROOF : The formula is checked using an easy and straightforward computation from the expression of the gaussian density ■

It is interesting to note that it is possible to define Gaussian probability on an infinite-dimensional vector space though it is not possible to define Lebesgue measure. However, in that chapter we just use finite-dimensional Gaussian probabilities. An interesting property of the Gaussian measure, which is crucial in this chapter is the following finite-dimensional version of the Girsanov theorem.

Theorem 2.20 *Let m_0 a probability measure on $\mathbb{R}^d$ and let $\mathcal{N}(\alpha, K)$ be a gaussian regular probability on $\mathbb{R}^d$. Let T a postive integer and $\mathcal{E}_T = (\mathbb{R}^d)^{\{0,\dots,T\}}$ the space of finite time traajectories in $\mathbb{R}^d$. Let w a gaussian random vector in $\mathcal{E}_T$ with law $m_0 \otimes \mathcal{N}(\alpha, K)^T$. Let ϕ and ψ two measurable applications of $\mathbb{R}^d$ into $\mathbb{R}^d$. Then we define the random vectors x and y in $\mathcal{E}$ by*

$$\begin{cases} x_0 = w_0 \\ x(t+1) = \phi[x(t)] + w(t+1) \end{cases}$$

$$\begin{cases} y_0 = w_0 \\ y(t+1) = \psi[y(t)] + w(t+1) \end{cases}$$

Let P and Q be the respective probability laws on $\mathcal{E}$ of x and y, then Q is absolutely continuous with respect to P and we have

$$\frac{dQ}{dP}(\eta) = \exp \sum_{t=0}^{T-1} \left\{ \begin{array}{l} -\frac{1}{2}\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\psi[(\eta(t)] - \phi[(\eta(t)]\} \\ +\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\eta(t+1) - \alpha - \phi[\eta(t)]\} \end{array} \right\} \tag{2.58}$$

PROOF : The proof is a recursion on T. It is easy to check (2.58) for $T = 1$. To reduce the expression let us write down

$$y_0^T = (y(0), ..., y(T)), \eta_0^T = (\eta(0), ..., \eta(T))$$

and

$$\Theta_T(\eta_0^T) = \sum_{t=0}^{T-1} \left\{ \begin{array}{l} -\frac{1}{2}\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\psi[(\eta(t)] - \phi[(\eta(t)]\} \\ +\{\psi[(\eta(t)] - \phi[(\eta(t)]\}^t K^{-1}\{\eta(t+1) - \alpha - \phi[\eta(t)]\} \end{array} \right\}$$

Suppose (2.58) is true up to T and let us compute the density of y up to $T+1$. Let h be a bounded continuous test function defined on $\mathcal{E}_{t+1}$. We have by conditioning with respect to y_0^T

$$\mathrm{E}\left[h(y(T+1), y_0^T)\right] = \int \mathrm{E}\left\{h(w(T+1) + \psi[\eta(T)], \eta_0^T)\right\} dQ(\eta_0^T)$$

where the expectation is taken with respect to $w(T+1)$, which is independent from y_0^T. Let us explicit the gaussian law $\mathcal{N}(\alpha, K)$ and use the recursion hypothesis:

$$\begin{array}{ll} \mathrm{E} & \left[h(y(T+1), y_0^T)\right] = \\ & C_K \int\int h(\omega + \psi[\eta(T)], \eta_0^T) \exp\left\{-\frac{1}{2}(\omega - \alpha)^t K^{-1}(\omega - \alpha)\right\} \\ & \exp \Theta_T(\eta_0^T) d\omega dP(\eta_0^T) \end{array}$$

where C_K is the classic normalization constant for the gausisan law. Then let us perform the translation $\varpi = \omega + \psi[\eta(T)]$, it gives

$$\begin{array}{ll} \mathrm{E} & \left[h(y(T+1), y_0^T)\right] = C_K \int\int h(\varpi, \eta_0^T) \\ & \exp\left\{-\frac{1}{2}(\varpi - \alpha - \psi[\eta(T)])^t K^{-1}(\varpi - \alpha - \psi[\eta(T)])\right\} \\ & \exp \Theta_T(\eta_0^T) d\varpi dP(\eta_0^T) \end{array}$$

To reduce notations let us write down $\zeta_T = \psi[\eta(T)] - \phi[\eta(T)]$, we have

$$\begin{array}{ll} \mathrm{E} & \left[h(y(T+1), y_0^T)\right] = C_K \int\int h(\varpi, \eta_0^T) \\ & \exp\left\{-\frac{1}{2}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)^t K^{-1}(\varpi - \alpha - \phi[\eta(T)] + \zeta_T)\right\} \\ & \exp \Theta_T(\eta_0^T) d\varpi dP(\eta_0^T) \end{array}$$

Let us develop the quadratic form under the exponential

$$\begin{array}{l} -\frac{1}{2}(\varpi-\alpha-\phi[\eta(T)]+\zeta_T)^t K^{-1}(\varpi-\alpha-\phi[\eta(T)]+\zeta_T) \\ =-\frac{1}{2}(\varpi-\alpha-\phi[\eta(T)])^t K^{-1}(\varpi-\alpha-\phi[\eta(T)]) \\ -\frac{1}{2}\zeta_T^t K^{-1}\zeta_T+\zeta_T^t K^{-1}(\varpi-\alpha-\phi[\eta(T)]) \end{array}$$

So we have

$$\begin{array}{l} \exp\left\{-\frac{1}{2}(\varpi-\alpha-\phi[\eta(T)]+\zeta_T)^t K^{-1}(\varpi-\alpha-\phi[\eta(T)]+\zeta_T)\right\} \\ =\exp\left\{\frac{1}{2}(\varpi-\alpha-\phi[\eta(T)])^t K^{-1}(\varpi-\alpha-\phi[\eta(T)])\right\} \\ .\exp\left\{-\frac{1}{2}\zeta_T^t K^{-1}\zeta_T+\zeta_T^t K^{-1}(\varpi-\alpha-\phi[\eta(T)])\right\} \end{array}$$

We obtain a product of two exponentials. The first one combines itself with $C_K d\varpi dP(\eta_O^T)$ to give $dP(\eta_O^{T+1})$; the second one combines itself with $\exp\Theta_T(\eta_0^T)$ to give $\exp\Theta_{T+1}(\eta_0^{T+1})$. So we get eventually

$$\mathrm{E}\left[h(y_0^{T+1})\right]=\int h(\eta_0^{T+1})\exp\Theta_{T+1}(\eta_0^{T+1})dP(\eta_0^{T+1})$$

■

2.6.3 Convergence of random variables

The definition of probability is based upon the law of large numbers (LLN). This last result may be roughly formulated as follows:
when (x_n) is an independant [5]sequence of random variables with the same probability law p with two first moments $c=\int xdp(x)$ and $k=\int x^2dp(x)$ then the sequence of empirical averages $\overline{x}_n=\frac{\sum_{k=1}^n x_k}{n}$ converges towards c.
That statement is not precise. The convergence may have several senses. Some useful convergence concepts in probability theory are the convergence in law, the convergence in probability and the almost sure convergence.
Let us recall their definitions:

Definition 2.14 *Let (x_n) and x be random variables on a probability space $(\Omega,\mathcal{A},P)$. The sequence of random variables (x_n) is said to*

- ***converge in law*** *to x if and only if for any continuous bounded function h, $E[h(x_N)]\rightarrow E[h(x)]$ [6]*
- ***converge in probability*** *to x if and only if*

$$\forall\epsilon>0, P(\mid x_n-x\mid\geq\epsilon)\rightarrow 0$$

[5]such a sequence is called an i.i.d. sequence

[6]An equivalent condition is the convergence of their characteristic functions (or Fourier transforms): $\forall t\in\mathbb{R},\mathrm{E}(\exp(itx_n))\rightarrow\mathrm{E}(\exp(itx))$

- ***converge almost surely*** *to* x *if and only if*

$$\exists N \subset \Omega \text{ with } P(N) = 0 \text{ such that } \forall \omega \notin N, x_n(\omega) \to x(\omega)$$

These definitions are stronger and stronger. Almost sure convergence implies convergence in probability which implies in turn convergence in law. Most mean-field computations of mean-field equations in random neural networks use the convergence of Fourier transforms through a Laplace limit integral ensuring convergence in law.
However, from the point of view of practitionners, almost sure convergence is more pleasant because a single realization of the sequence (X_n) allows to check the convergence. To check the weaker convergence statements, a lot of realizations of the sequence are necessary.
Let us return to the large number law: the convergence in probability of the sequence $(\overline{x}_n)$ is specially easy to show since $\mathrm{E}(\overline{x_n}) = c$ and $\mathrm{Var}(\overline{x}_n) = \frac{k-c^2}{n}$. Then one has just to write the Bienaymé-Tchebychev inequality

$$P(\mid \overline{x}_n - c \mid \geq \epsilon) \leq \frac{k - c^2}{n\epsilon^2}$$

But this convergence is not strong enough to show the almost sure convergence (the so-called **strong** large number law).

2.6.4 Large deviation principle

Cramer's theorem

One way to obtain the strong law is to show that the convergence in probability occurs much faster than it appears from Bienaymé-Tchebychev inequality.
Actually the following theorem was obtained by Cramer in the late 30's:

Theorem 2.21 *Let* (x_n) *sequence of i.i.d. random variables with probability law* μ *such that* $\int \xi d\mu(\xi) = \overline{\theta}$. *Then we have*

$$\forall a > \xi, \ \frac{1}{n} \log P(\overline{x}_n > a) \to -I(a)$$

where

$$I(a) = \max_{p \in \mathbb{R}} pa - E[\exp(px)]$$

This theorem can be extended to more general settings. It is the subject of **large deviation** theory. Let us first consider the case of finite-dimensional random vectors [22]
The following proposition is easy to prove:

Proposition 2.22 *Let μ a probability law on $\mathbb{R}^d$ such that for all $p \in \mathbb{R}^d$, $\Lambda(p) = \log E[\exp(p^t x)]$ exists. The function Λ is called the **log-generating function** of μ. We define its **Legendre transform** Λ^* on $\mathbb{R}^d$ as follows:*

$$\Lambda^*(a) = \sup_{p \in \mathbb{R}^d} [(p^t a) - \Lambda(p)]$$

then

a) Λ^* is a convex function (with ∞ as a possible value)

b) $\forall a \in \mathbb{R}^d, \Lambda^*(a) \geq 0$

c) $a = \int \xi d\mu(\xi) \Leftrightarrow \Lambda^*(a) = 0$

PROOF : a) is straightforward, since the supremum of convex functions is convex
b) comes from Jensen's inequality.
c) comes from $\Lambda(0) = 1$

■

Then we can state the Cramer's theorem for i.i.d. sequence of finite-dimensional random vectors:

Theorem 2.23 ***Cramer's theorem:***
Let (x_n) be a sequence of i.i.d. random vectors with a probability distribution μ according to the assumption and the notations of the previous proposition. Then for any Borel subset B of $\mathbb{R}^d$, we have

$$-\inf_{a \in B^o} \Lambda^*(a) \leq \frac{1}{n} \underline{\lim}_{n \to \infty} \log[\mathbb{P}(\overline{x}_n \in B^o)] \leq \frac{1}{n} \overline{\lim}_{n \to \infty} \log[\mathbb{P}(\overline{x}_n \in \overline{B})] \leq -\inf_{a \in \overline{B}} \Lambda^*(a) \tag{2.59}$$

where B^o is the interior set of B (the greatest open subset of B) and $\overline{B}$ is the closure of B (the smallest closed extension of B).

A consequence of Cramer's theorem is that for any closed subset F in $\mathbb{R}^d$ such that $\inf_{a \in \overline{B}} \Lambda^*(a) > 0$, $\mathbb{P}(\overline{X}_n \in F)$ goes to 0 exponentially fast when $n \to \infty$ and that the rate of convergence depends only on the value of Λ^* at the point of F where Λ^* reaches its minimum. This point is called the

dominating point. For regular probability distributions where Λ^* is strictly convex, defined and continuous around $\overline{\theta} = \mathrm{E}(x)$, the exponential decay of finite deviations from the expectation (large deviations) and the strong law of large numbers are easy consequences.

Large deviation principle in an abstract setting

The convergence with an exponential rate is a general situation, which is characterized in the following general definitions:

Definition 2.15 *Let $\mathcal{E}$ be a Polish space and I be a lower semi-continuous function of $\mathcal{E}$ into $[0, \infty]$. I is called a **rate function**. If I posesses the property of compact level set, i.e.*

$$\forall \epsilon > 0, \{x \in \mathcal{E} \text{ such that } I(x) \leq \epsilon\} \text{ is compact}$$

*then I is called a **good rate function**.*

Definition 2.16 *Given a rate function I on a Polish space $\mathcal{F}$ and a sequence of probability measures Q_n on $\mathcal{F}$, if for any Borel subset B of $\mathcal{F}$,*

- *(Q_n) satisfies the **large deviation minoration on open sets** if*

$$\forall O, \text{ open set in } \mathcal{F}, -\inf_{\xi \in O} I(\xi) \leq \frac{1}{n} \underline{\lim}_{n \to \infty} \log[Q_n(O)] \tag{2.60}$$

- *(Q_n) satisfies the **large deviation majoration on compact sets** if*

$$\forall K, \text{ compact set in } \mathcal{F}, \frac{1}{n} \overline{\lim}_{n \to \infty} \log[Q_n(K))] \leq -\inf_{\xi \in K} I(x) \tag{2.61}$$

- *(Q_n) satisfies the **large deviation majoration on closed sets** if*

$$\forall C, \text{ closed set in } \mathcal{F}, \frac{1}{n} \overline{\lim}_{n \to \infty} \log[Q_n(C))] \leq -\inf_{\xi \in C} I(x) \tag{2.62}$$

- *If (Q_n) checks the large deviation minoration for open sets and the large deviation majoration for compact sets we say that (Q_n) satisfies the **large deviation principle (LDP)** with rate function I.*

- *If (Q_n) checks the large deviation minoration for open sets and the large deviation majoration for closed sets we say that (Q_n) satisfies the **full large deviation principle** with rate function I.*

- *(Q_n) is said **tight** if forall $\epsilon > 0$, it exists a compact subset K of $\mathcal{F}$ such that $Q_n({}^cK) < \epsilon$. If (Q_n) is tight and checks a LDP, it satisfies the full LDP for the same rate function.*

The same definitions stand for a sequence of random elements in $\mathcal{F}$ if the sequence of their probability laws checks the respective majorations.
A simpler way to state that (Q_n) satisfy the full large deviation principle with rate function I is to write that

$$-\inf_{\xi\in B^o} I(\xi) \leq \frac{1}{n}\underline{\lim}_{n\to\infty} \log[Q_n(B)] \leq \frac{1}{n}\overline{\lim}_{n\to\infty} \log[Q_n(B))] \leq -\inf_{\xi\in\overline{B}} I(x) \tag{2.63}$$

Actually, the scope of Cramer's theorem may be widely extended and a full large deviation principle is checked for the empirical mean of any i.i.d. random sequence in a Polish space under mild assumptions on the existence of the log-generating function [13]. The rate function of this LDP is the Legendre transform of the log-generating function.

Varadhan theorem and Laplace principle

An equivalent functional formulation of the full large deviation principle is due to Varadhan and is called by Dupuis and Ellis the **Laplace principle** ([15]).

Definition 2.17 *Let I be a good rate function on the Polish space $\mathcal{F}$. The random sequence (x_n) in $\mathcal{F}$ is said to satisfy the **Laplace principle** with rate function I if for any continuous bounded function h on $\mathcal{E}$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \log E\{\exp[-nh(x_n)]\} = -\inf_{\xi\in\mathcal{F}}\{h(\xi) + I(\xi)\}$$

This approach is called by the authors of ([15]) the **weak convergence approach** to the theory of large deviations. The equivalence of the two approaches (Laplace principle for good rate functions and full large deviation principle with good rate functions) are expressed in a theorem of Varadhan and its converse. Their proofs are in ([15]). Handling continuous bounded test functions may be more practical than dealing with open and closed sets. In particular, it is very easy to show the following transfer theorem for the LDP principle when the law is changed.

Theorem 2.24 *Let P_n and Q_n two sequences of probability measures on the Polish space $\mathcal{F}$, let I be a good rate function on F and let Γ a continuous function on F such that*

(a) $Q_n << P_n$ *and* $\frac{dQ_n}{dP_n}(\xi) = \exp n\Gamma(\xi)$,

(b) (P_n) *satisfies a full large deviation principle with rate function* I,

(c) $I - \Gamma$ *is a good rate function,*

then (Q_n) *satisfies a full large deviation principle with rate function* $I - \Gamma$.

PROOF OF THEOREM: Using the weak large deviation approach and the strong hypothesis of the theorem, the proof is quite formal. Let h be any continuous bounded test function on $\mathcal{F}$, from hypothesis (c) and ■

2.6.5 Convergence of random measures

Let us have a second look at the law of large numbers. Since this law claims the convergence on the sequence of empirical averages $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ over any bounded continuous test function f we are lead to consider the empirical measure of a sample.

Definition 2.18 *Let* $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^{nd}$ *a sequence of* n *vectors of* $\mathbb{R}^d$. *We associate to* ξ *the following probability measure* $\mu_\xi \in \mathcal{P}(\mathbb{R}^d)$

$$\mu_\xi = \frac{1}{n}\sum_{k=1}^{n} \delta_{\xi_k}$$

$\mu_x i$ *is called the* ***empirical measure*** *associated to* ξ.

This definition says that if A is a Borel subset of $\mathcal{F}$ then $\mu_N(x)(A)$ is the fraction of neurons which state trajectory belong to A. More practically, if ϕ is any test continuous function on $\mathcal{E}$, it says that

$$\int_{\mathcal{E}} \phi(\eta) d\mu_N(u)(\eta) = \frac{1}{N}\sum_{i=1}^{N} \phi(u_i)$$

With this definition, the convergence for each continuous bounded test function f of $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ towards $\int f(\xi)d\mu(\xi)$ is exactly the narrow convergence of the sequence μ_{x_n} towards μ.
The set $\mathcal{P}(\mathbb{R}^d)$ of probability measure on $\mathbb{R}^d$ is a convex subset of the functional vector space $\mathcal{M}^1(\mathbb{R}^d)$ of bounded measures on $\mathbb{R}^d$. We endow $\mathcal{P}(\mathbb{R}^d)$ with the **narrow topology** for which $\mu_n \to \mu$ if and only if for all continuous and bounded test function $f \in \mathcal{C}^b(\mathbb{R}^d), \int f d\mu_n \to \int f d\mu$. $\mathcal{P}(\mathbb{R}^d)$ is a Polish state for this topology.

So instead of considering the random variable x which takes its values in $\mathbb{R}^d$, we consider the random variable δ_ξ which takes its values in the Polish space $\mathcal{P}(\mathbb{R}^d)$. If (x_k) is an i.i.d. sequence in $\mathbb{R}^d$ with probability law μ, then δ_{x_i} is an i.i.d. sequence in $\mathcal{P}(\mathbb{R}^d)$ and its empirical mean is just $\mu_{(x_1,\ldots,x_n)}$ the empirical measure of an i.i.d. sample of size n. That means that Cramer's theorem extension to Polish spaces may be applied. This theorem is known as **Sanov theorem**.
Let us first recall the definition of the relative entropy with respect to a probability measure μ on $\mathbb{R}^d$.

Definition 2.19 *Let μ be a probability measure on $\mathbb{R}^d$. We define a convex function $\nu \in \mathcal{P}(\mathbb{R}^d) \to I(\nu,\mu) \in \mathbb{R}$ by:*

$$\begin{cases} I(\nu,\mu) = \int \log \frac{d\nu}{d\mu}(\xi) d\nu(\xi) \\ I(\nu,\mu) = \infty \ \textit{else} \end{cases} \tag{2.64}$$

*This function is called the **relative entropy** with respect to μ*

then we may state the Sanov theorem [16], [13]

Theorem 2.25 *The sequence of empirical measure μ_n which are associated to size n i.i.d. sample of a probability law on $\mathbb{R}^d$ satisfy a full LDP with the **relative entropy** with respect to μ as the rate function.*

Bibliography

[1] S. Amari. Characteristics of random nets of analog-like elements. *IEEE Trans. Syst. Man. Cyb.*, SMC-2(N5), 1972.

[2] S. Amari, K. Yosida, and K.I. Kakutani. A mathematical foundation for statistical neurodynamics. *Siam J. Appl. Math.*, 33(1):95–126, 1977.

[3] D.J. Amit and J. Brunel. Dynamics of a recurrent network of spiking neurons before and following learning. *Network: Computation in Neural Systems*, 8:373–404, 1997.

[4] D.J. Amit and J. Brunel. Model of global spontaneous activity and local structured delay activity during delay periods in the cerebral cortex. *Cerebral Cortex*, 7:237–252, 1997.

[5] G. Ben Arous and A. Guionnet. Large deviations for langevin spin glass dynamics. *Probability Theory and Related Fields*, 102:455–509, 1995.

[6] N. Brunel. Dynamics of sparsely connected networks of excitatory and inhibitory spiking neurons. *Journal of Computational Neuroscience*, 8:183–208, 2000.

[7] N. Brunel and V. Hakim. Fast global oscillations in networks of integrate-and-fire neurons with low firing rates. *Neural Computation*, 11:1621–1671, 1999.

[8] G. Buzsaki and J.J. Chrobak. Temporal structure in spatially organized neuronal ensembles. a role for interneuronal networks. *Current Opinion in Neurobiology*, 5:504, 1995.

[9] B. Cessac. Increase in complexity in random neural networks. *Journal de Physique I*, 5:409–432, 1995.

[10] B. Cessac, B. Doyon, M. Quoy, and M. Samuelides. Mean-field equations, bifurcation map and route to chaos in discrete time neural networks. *Physica D*, 74:24–44, 1994.

[11] E. Daucé, O. Moynot, 0.Pinaud, and M. Samuelides. Mean field theory and synchronization in random recurrent neural networks. *Neural Processing Letters*, 14:115–126, 2001.

[12] E. Daucé, M. Quoy, B. Cessac, B. Doyon, and M. Samuelides. Self-organization and dynamics reduction in recurrent networks: stimulus presentation and learning. *Neural Networks*, 11:521–533, 1998.

[13] A. Dembo and O. Zeitouni. *Large Deviations techniques*. Jones Bartlett publishers, 1993.

[14] B. Derrida and Y. Pommeau. *Europhys. lett.*, 1:45–59, 1986.

[15] P. Dupuis and R.S. Ellis. *A weak convergence approach to the theory of large deviations*. Wiley, 1997.

[16] Richard S. Ellis. *Entropy, Large Deviations and Statistical Mechanics*. Springer Verlag, 1985.

[17] W. Gerstner and W.M. Kistler. *Spiking neuron models, single neurons, populations, plasticity*. Pure and Applied Mathematics Series. Cambridge University Press, 2002.

[18] Gray. Synchronous oscillations in neuronal systems: Mechanisms and functions. *Journal of computational Neuroscience*, 7:334–338, 1994.

[19] A. Guillot and E.Daucé. *Approches dynamiques de la cognition artificielle*. Hermès, Paris, 2002.

[20] D. Hansel and H. Sompolinsky. Chaos and synchrony in a model of a hypercolumn in visual cortex. *J. Comp. Neurosc.*, 3:7–34, 1996.

[21] J. J. Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proc. Nat. Acad. Sci.*, 79:2554–2558, 1982.

[22] O. Kallenberg. *Foundations of Modern Probability*. Probabilities and its applications. Springer-Verlag, 2002.

[23] N.G. Van Kampen. *Stochastic processes in Physics and Chemistry*. North Holland, 1992.

[24] Molgedey L., Schuchardt J., and Schuster H.G. Supressing chaos in neural networks by noise. *Physical Review Letters*, 69(26):3717–3719, 1992.

[25] W. S. McCulloch and W. Pitts. A logical calculus of ideas immanent in nervous activity. *Bulletin of Mathematical Biophysics*, 5:115–133, 1943.

[26] O. Moynot. *Etude mathématique de la dynamique des réseaux aléatoires récurrents*. PhD thesis, Université Paul Sabatier, Toulouse, 1999.

[27] O.Moynot and M.Samuelides. Large deviations and mean-field theory for asymetric random recurrent neural networks. *Probability Theory and Related Fields*, 123(1):41–75, 2002.

[28] R. Ritz and T.J. Sejnowski. Synchronous oscillatory activity in sensory systems: new vistas on mechanisms. *Current Opinion in Neurobiology*, 7(4):536–546, 1997.

[29] W. Singer and C. M. Gray. Visual feature integration and the temporal correlation hypothesis. *Annual review in Neuroscience*, 5, 1995.

[30] C.A. Skarda and W.J. Freeman. *Chaos and the new science of the brain*, volume 1-2, pages 275–285. In "Concepts in Neuroscience", World Scientfic Publishing Company, 1990.

[31] H. Sompolinsky, A. Crisanti, and H.J. Sommers. Chaos in random neural networks. *Phys. Rev. Lett.*, 61:259–262, 1988.

[32] A.S. Sznitman. Non linear reflecting diffusion process and the propagation of chaos and fluctuations associated. *Journal of Functional Analysis*, 56(3):311–336, 1984.