Singleton Representation of Fuzzy Set for Computing Fuzzy Model Response for Fuzzy Inputs

Karina Murawko-Wiśniewska[1], Andrzej Piegat[1]

[1] Szczecin University of Technology, Żołnierska 49, 71-210 Szczecin, Poland
{KMurawko, APiegat}@wi.ps.pl

Abstract. Classical fuzzy model computes a crisp response for crisp inputs. This paper presents a method for computing fuzzy model response for fuzzy inputs. The method is based on singleton representation of a fuzzy set and it allows to obtain fuzzy response for fuzzy inputs. The presented method is compared with alternative approaches: Zadeh's possibilistic method and method based on similarity measure. The validity of the proposed method is illustrated with experimental results (in comparison with extension principle results).

1 Introduction

There are different types of data in the real world: precise (numerical) data and uncertain data. Numerical data, collected with a precise measuring instrument, can be analyzed and processed by numerical mathematical methods. Uncertainty of the data can be caused by e.g. imprecision of sensor (human notions) or an attribute that is not quantifiable. Fuzzy set concept, introduced by Zadeh in 1965 [14], is a useful tool for a formal representation of uncertain information. Fuzzy set models the value of a linguistic variable. In [17] Zadeh distinguishes four cases which underlie the use of linguistic variables: bounded ability of sensory organs to resolve and store detail information (e.g. brain), numerical information may not be available, an attribute is not quantifiable, there is a tolerance for imprecision.

Table 1. Illustrative example: a mixed data set which describes flats for sale

x_1 – price [€]	x_2 – total area [m^2]	x_3 – building age [years]	x_4 – flat standard	x_5 – location attractiveness
140000	31	26	high	average
265000	87	76	low	high
160000	42	23	average	low
205000	63	7	very high	average

Illustrative example: a data set which describes flats for sale, each row contains values of five attributes. Three of them are numerical variables (price, total area, building age); others are linguistic variables (flat standard, location attractiveness). Values of the linguistic variables (x_4, x_5) can not be measured precisely. Assigning each of those values to one precise (numerical) value or removing variables x_4, x_5 would be an oversimplification which would cause lost of information. Developing methods which process mixed data (precise and uncertain) would enable the use of the whole available data.

Major applications of fuzzy set theory are fuzzy modeling and control, their development started with papers of Zadeh [15], Mamdani and Assilian [6]. Classical fuzzy model [11], [12] consists of three main blocks: fuzzification, inference and deffuzzification (Fig. 1.). It computes a crisp (numerical) output of the model for crisp inputs. Therefore it is not possible to compute an output of classical fuzzy model if any input is uncertain (fuzzy). Despite the dynamic development of fuzzy modeling research field, only a few papers concerning fuzzy models and controllers with fuzzy inputs have been published. In [1], the design of inverse controller for fuzzy interval systems is exploited. In [10], a linguistic approach to the design of fuzzy granular models is concerned. In [5], the theory and design of interval type-2 fuzzy logic systems are presented.

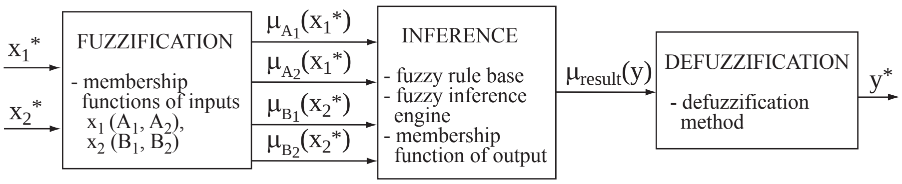

Fig. 1. Structure of an exemplary fuzzy model with two inputs and a single output [11]

The purpose of this study is to develop a method of computing fuzzy model response for fuzzy inputs based on singleton representation of a fuzzy set. The key idea of our approach is to compute fuzzy model output for each singleton representing input fuzzy set and combine the results into an output fuzzy set. The main advantage of this approach is to enable computing the output of fuzzy model both for fuzzy inputs and mixed (numerical and fuzzy) inputs (using the classical fuzzy model without modification). The paper is organized as follows. In section 2, alternative approaches to the problem are presented. In section 3, we apply the singleton representation to computing fuzzy model response for fuzzy inputs. Section 4 shows results of experiments. Finally, conclusions are given in section 5.

2 A review of methods of computing fuzzy model output for fuzzy input

In the approach proposed in [16]: Zadeh's possibilistic method, used for computing with linguistic variables, fuzzy set A is compared with fuzzy sets describing the linguistic variable (A_i) and for each set μ_i is given by

$$\mu_i = \sup(A_i \cap A)\,. \tag{1}$$

This method describes with a single crisp value (maximal possibility) how much set A is similar to set A_i. The use of the method to compute fuzzy model output for fuzzy inputs results in the crisp output to be received.

The next approach involves similarity measures. Many similarity measures of fuzzy sets have been proposed in the literature [9], [2], [13], [3]. The commonly used similarity measure, proposed by Pappis and Karacapilidis [9], for two fuzzy sets A and B (with continuous membership functions) is defined by

$$S(A,B)=\frac{|A\cap B|}{|A\cup B|}=\frac{\int_{x\min}^{x\max}\min(\mu_A(x),\mu_B(x))dx}{\int_{x\min}^{x\max}\max(\mu_A(x),\mu_B(x))dx}, \tag{2}$$

where $|A|$ – denotes cardinality of fuzzy set A; *xmin*, *xmax* are the boundaries of universe of discourse X. The use of similarity measures of fuzzy sets for computing response of fuzzy model has been presented in [7]. Similarity measure is used in fuzzification to specify similarity between fuzzy value of input and the fuzzy sets describing the fuzzy variable. The model response is non-fuzzy (crisp) value and it depends on the chosen similarity measure.

In the presented approaches fuzzy model response for fuzzy inputs is a crisp value. It is not what one would expect: though input value is fuzzy (uncertain), the output is non-fuzzy.

3 Using Singleton Representation of Fuzzy Set for Computing Fuzzy Model Response for Fuzzy Input

There are different representations of fuzzy set in the literature. For example, in discrete universe of discourse $X=\{x_1,\ldots,x_n\}$ fuzzy set $A\subseteq X$ can be represented by

$$A=\frac{\mu_A(x_1)}{x_1}+\frac{\mu_A(x_2)}{x_2}+\ldots+\frac{\mu_A(x_n)}{x_n}=\sum_{i=1}^{n}\frac{\mu_A(x_i)}{x_i}. \tag{3}$$

It is the singleton (vertical) representation of a fuzzy set [11], [12,] [4]. The fuzzy set with continuous membership function (X – continuous universe of discourse) is given by

$$A=\int_X\frac{\mu_A(x)}{x}. \tag{4}$$

For any input fuzzy set the fuzzy model response can be computed using the singleton representation. First, fuzzy sets is represented by set of singletons (3), therefore fuzzy sets with continuous membership functions are discretized. The idea of discretizing the continuous membership functions of fuzzy sets and the use of discrete representation of fuzzy sets in fuzzy arithmetical operations is presented in [4]. The range of singletons for exemplary fuzzy set A is shown at Fig. 2. The singleton s_3,

$\mu(s_3)$=1, has the widest representation range, whereas boundary singletons s_1 and s_5 ($\mu(s_1)$= $\mu(s_5)$=0) have the range reduced to a point.

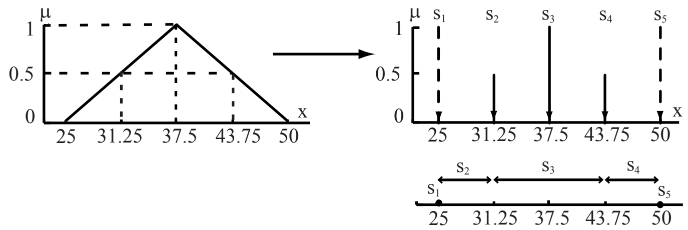

Fig. 2. Fuzzy set *A*="about 37.5" in continuous and singleton representation

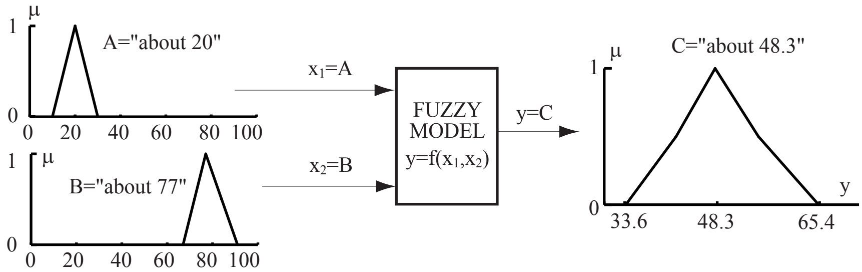

Fig. 3. Computing fuzzy model response for two input fuzzy values

For each singleton of input fuzzy set the model response is computed. Then the output fuzzy set is created with the use of extension principle: the fuzzy model is represented by a mapping $f: X \to Y$ and for any fuzzy set $A \subseteq X$ fuzzy set $B=f(A)$ is defined by

$$B = f(A) = \{(y, \mu_B(y)) \mid y = f(x), x \in X\}, \tag{5}$$

where

$$\mu_B(y) = \begin{cases} \sup\limits_{x \in X, y=f(x)} \mu_A(x) \\ 0, \quad \text{else} \end{cases}. \tag{6}$$

If X is the Cartesian product $X_1 \times X_2 \times ... \times X_n$, the fuzzy model is represented by a mapping $f: X_1 \times X_2 \times ... \times X_n \to Y$, for any fuzzy sets $A_1 \subseteq X_1$, $A_2 \subseteq X_2$, ..., $A_n \subseteq X_n$, fuzzy set $B=f(A_1, A_2 ..., A_n)$ is defined by

$$B = f(A_1,..., A_n) = \{(y, \mu_B(y)) \mid y = f(x_1,..., x_n), (x_1,..., x_n) \in X\}, \tag{7}$$

where

$$\mu_B(y) = \begin{cases} \sup\limits_{(x_1,...,x_n) \in X, y=f(x_1,...,x_n)} \min\{\mu_{A_1}(x_1),..., \mu_{A_n}(x_n)\} \\ 0, \quad \text{else} \end{cases}. \tag{8}$$

In (8) minimum operation can be replaced by other t-norm (e.g. algebraic product).

Example: fuzzy model with two inputs x_1 and x_2 is given. The values of inputs can not be measured precisely, though the expert describes them with fuzzy sets: A="about 20" and B="about 77". To compute fuzzy model response, input fuzzy sets $x_1=A$ and $x_2=B$ (with continuous membership functions) are replaced by singleton representation. Next, for each pair of singletons of A and B an output of the model is computed. For each output singleton its membership function value μ_C is computed (where fuzzy set $C=f(A,B)$), and if the output singletons for two pairs of input singletons are the same the maximum membership is chosen according to (9). The result of the computation is fuzzy set C="about 48.3" (Fig. 3.)

4 Experiments

In this section we present the results of using our method to compute fuzzy model response for fuzzy inputs. The synthetic data set that contains fuzzy sets (Fig. 7) is used as inputs in 10 experiments. The computation is done with two classical fuzzy models designed using the known functions f_1 , f_2. The known function are used to enable comparison of the results of the proposed method. The models have two inputs (x_1, x_2) and an output (y). Fuzzy sets of inputs and outputs of both models: A_i, B_j, C_k ($i,j,$ =1,..,5; model 1: k=1,...,10; model 2: k=1,...,16), are shown at Fig. 5 (A_i for x_1 and B_j for x_2). The rules used in the models are given in the following form: "IF $x_1=A_i$ AND $x_2=B_j$ THEN $y=C_k$". The rule base of model 1 and model 2 is given in Table 1. The inference engine is MAX-MIN type and defuzzification is done by height method.

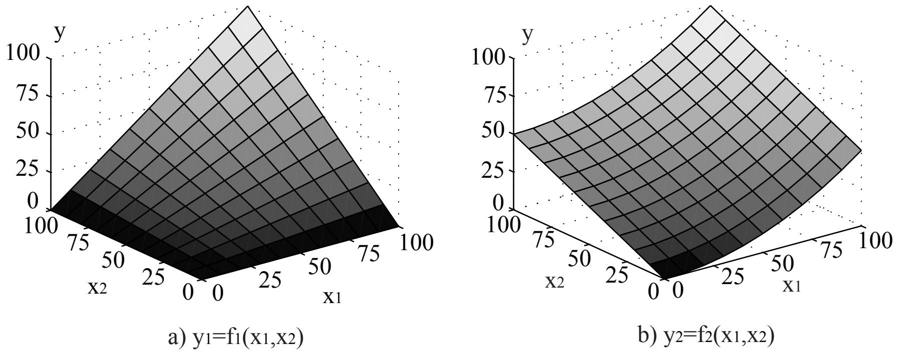

Fig. 4. The surfaces of the mappings used in experiments

Fig. 4. shows the surfaces of the functions used in experiments, function f_1 is used in experiments no. 1-7, 9 and function f_2 is used in experiments no. 8 and 10. The functions f_1 and f_2 are defined as follows

$$y_1 = f_1(x_1, x_2) = \frac{x_1 \cdot x_2}{100} \qquad \textbf{(9)}$$

and

$$y_2 = f_2(x_1, x_2) = \left(\frac{x_1^2}{100} + x_2 \right) / 2 \ . \qquad (\mathbf{10})$$

Based on functions f_1 and f_2 two fuzzy models are designed, their surfaces are shown in Fig. 6. The values of x_1, x_2, y_1, y_2 belong to the interval <0;100>.

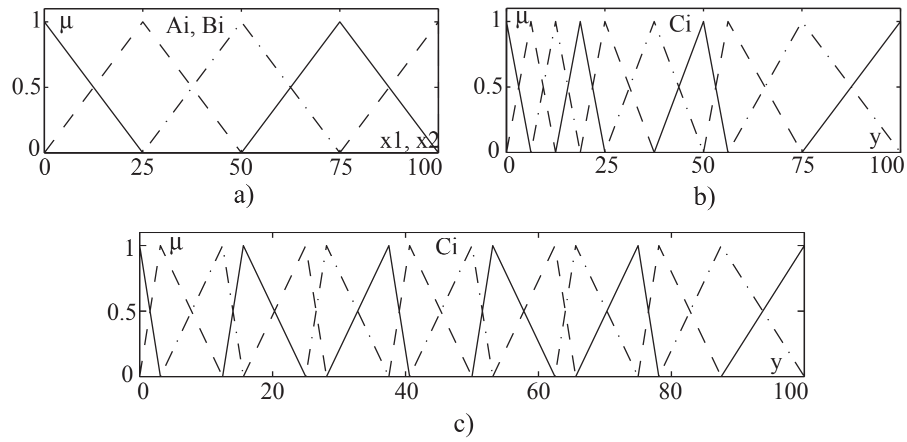

Fig. 5. Fuzzy sets of inputs (a), output of model 1 (b) and output of model 2 (c)

Table 2. The rule base of model 1 and model 2

MODEL 1					
x_2 \ x_1	A_1	A_2	A_3	A_4	A_5
B_1	C_1	C_1	C_1	C_1	C_1
B_2	C_1	C_2	C_3	C_4	C_5
B_3	C_1	C_3	C_5	C_6	C_7
B_4	C_1	C_4	C_6	C_8	C_9
B_5	C_1	C_5	C_7	C_9	C_{10}

MODEL 2					
x_2 \ x_1	A_1	A_2	A_3	A_4	A_5
B_1	C_1	C_2	C_3	C_6	C_9
B_2	C_3	C_4	C_5	C_8	C_{11}
B_3	C_5	C_6	C_7	C_{10}	C_{13}
B_4	C_7	C_8	C_9	C_{12}	C_{15}
B_5	C_9	C_{10}	C_{11}	C_{14}	C_{16}

In each experiment the input values are two fuzzy sets: $x_1=F_1$ and $x_2=F_2$ (Fig. 7), their singleton representations are given in Table 2. Firstly extension principle is used for computing fuzzy sets $F_3=f_1(F_1,F_2)$, $F_4=f_2(F_1,F_2)$, which are reference results (results of other methods are compared with them). Then the singleton representation of input fuzzy sets F_1 and F_2 is used for computing fuzzy model response (model 1 and 2): Y_1, Y_2 (fuzzy values). Finally Zadeh's possibilistic method and similarity measure method are used for computing fuzzy model response in form of a singleton (crisp value, accordingly z and s).

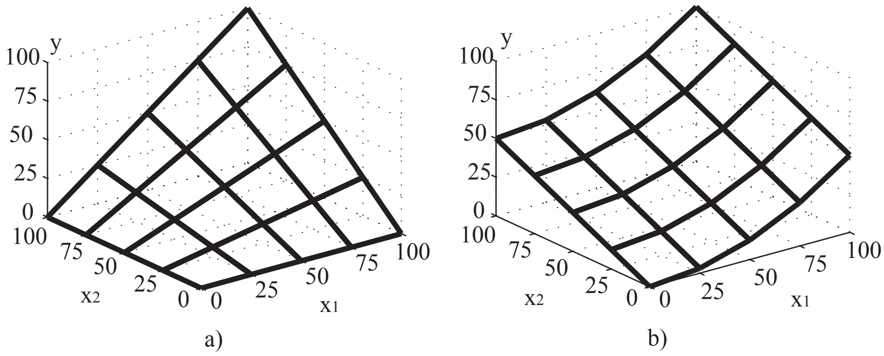

Fig. 6. The surfaces of fuzzy models: (a) – model 1, (b) – model 2

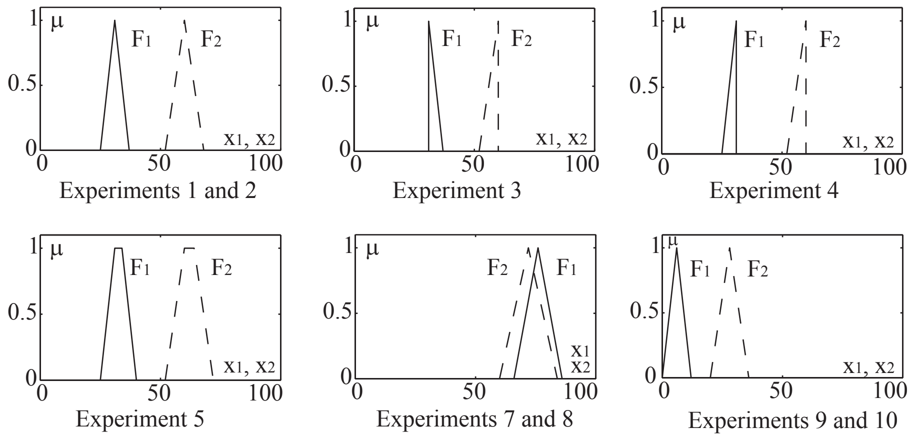

Fig. 7. Input fuzzy sets x_1=F_1 and x_2=F_2; in experiment no. 6: x_1=A_3 and x_2=B_2

Table 3. Singleton representation of input fuzzy sets F_1 and F_2

Experiment No.	Singleton Representation	
	F_1	F_2
1	$F_1=\frac{0}{25}+\frac{0.5}{28}+\frac{1}{31}+\frac{0.5}{34}+\frac{0}{37}$	$F_2=\frac{0}{52}+\frac{0.5}{56}+\frac{1}{60}+\frac{0.5}{64}+\frac{0}{68}$
2	$F_1=\frac{0}{25}+\frac{1}{31}+\frac{0}{37}$	$F_2=\frac{0}{52}+\frac{1}{60}+\frac{0}{68}$
3	$F_1=\frac{1}{31}+\frac{0.5}{34}+\frac{0}{37}$	$F_2=\frac{0}{52}+\frac{0.5}{56}+\frac{1}{60}$
4	$F_1=\frac{0}{25}+\frac{0.5}{28}+\frac{1}{31}$	$F_2=\frac{0}{52}+\frac{0.5}{56}+\frac{1}{60}$
5	$F_1=\frac{0}{25}+\frac{0.5}{28}+\frac{1}{31}+\frac{1}{34}+\frac{0.5}{37}+\frac{0}{40}$	$F_2=\frac{0}{52}+\frac{0.5}{56}+\frac{1}{60}+\frac{1}{64}+\frac{0.5}{68}+\frac{0}{72}$
6	$F_1=\frac{0}{25}+\frac{0.5}{37.5}+\frac{1}{50}+\frac{0.5}{62.5}+\frac{0}{75}$	$F_2=\frac{0}{0}+\frac{0.5}{12.5}+\frac{1}{25}+\frac{0.5}{37.5}+\frac{0}{50}$
7, 8	$F_1=\frac{0}{66}+\frac{0.5}{71}+\frac{1}{76}+\frac{0.5}{81}+\frac{0}{86}$	$F_2=\frac{0}{60}+\frac{0.5}{66}+\frac{1}{72}+\frac{0.5}{78}+\frac{0}{84}$
9, 10	$F_1=\frac{0}{0}+\frac{0.5}{3}+\frac{1}{6}+\frac{0.5}{9}+\frac{0}{12}$	$F_2=\frac{0}{20}+\frac{0.5}{24}+\frac{1}{28}+\frac{0.5}{32}+\frac{0}{36}$

Table 4. Comparison of the results: modal values of fuzzy sets computed with extension principle (EP) and computed by fuzzy model with singleton representation (SFM); crisp values of fuzzy model response computed using possibilistic method (MP) and similarity measure method (SM)

No.	EP (F_3, F_4)	SFM (Y_1, Y_2)	MP (z)	SM (s)
1	18.6	20.3	21.4	19.9
2	18.6	20.3	21.4	19.9
3	18.6	20.3	21.7	21.5
4	18.6	20.3	19.8	18.5
5	<18.6;21.8>	<20.3;22.38>	22.1	20.53
6	12.5	12.5	8.9	12.2
7	54.7	54.6	55.3	56.7
8	64.9	65.4	66.9	68.1
9	1.7	2.4	3.0	2.3
10	14.2	15.8	15.9	15.2

Singleton computed with the use of similarity measure method in most of the experiments is proximal to the modal value of fuzzy sets computed with extension principle then singleton computed with the use of Zadeh’s possibilistic method. Both methods’ results are crisp values: the information about uncertainty is lost in computations.

On the contrary, computing fuzzy model response with singleton representation preserves data uncertainty. The shapes of resulting fuzzy sets F_3/F_4 and Y_1/Y_2 are also similar (Fig. 8.). Results of the proposed method depend on the number of singletons in the representation and precision of the model. The best performance (experiment no. 6) is obtained for input fuzzy sets with modal values in the nodes of the model (in nodes the error of both fuzzy models used in experiments is zero).

5 Conclusions

In this paper we proposed a method for computing fuzzy model response for fuzzy inputs. The method is consistent with extension principle, where the model is represented by a mapping *f*. It can be used for computing fuzzy model response for any number of input fuzzy sets and also for mixed input data (fuzzy and crisp) and no modification of the model is necessary. If any input is a fuzzy value the model response is a fuzzy set. The experimental results demonstrated that usage of this method provides fuzzy output for fuzzy inputs, analogical to results of computing fuzzy values using extension principle. The shapes of resulting fuzzy sets for both computing methods are also similar. In real situations, when data are often mixed types, the proposed method can be useful for computing fuzzy model response for mixed inputs. In the future, we plan to focus on design of fuzzy models based on mixed input-output data (crisp and fuzzy).

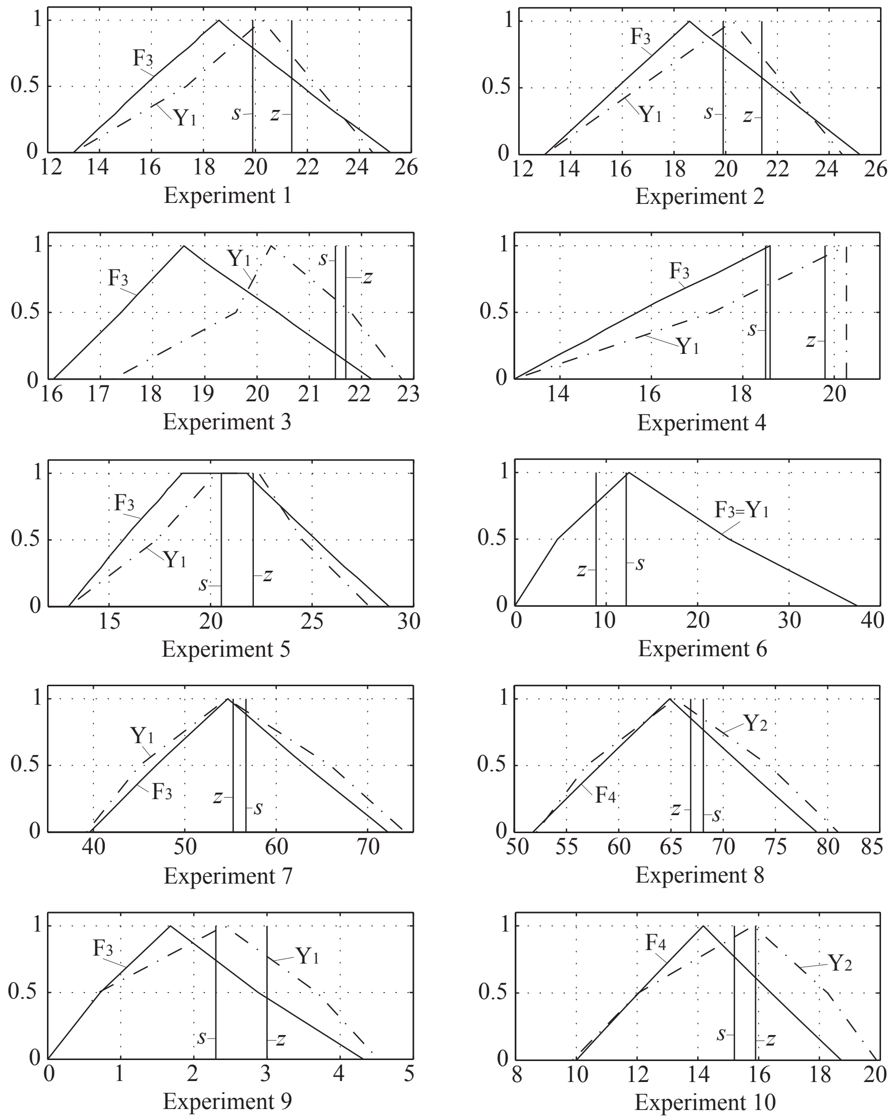

Fig. 8. The graphical presentation of the results: (F_3, F_4) – extension principle result, (Y_1, Y_2) – fuzzy model response, s – the result of method based on similarity, z – the result of possibilistic method

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