THE UNIVERSITY OF SYDNEY

MATH 3901

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Metric Spaces 2006

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**Tutorial 11**

**1.** Let $X$ be a set and let $L^2(X, \mu_c)$ be the Hilbert space defined in Tutorial 10, with underlying vector space the set of all functions $f : X \to \mathbb{C}$ such that

$$n(f) = \sup\{\Sigma_{x \in F} |f(x)|^2 \mid F \subseteq X,\ F\ finite\}$$

is finite and with inner product given by

$$b(f, g) = \sup\{\Sigma_{x \in F} f(x)\overline{g(x)} \mid F \subseteq X,\ F\ finite\}.$$

Show that the subset consisting of all functions $\delta_x$ such that $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \neq x$ is an orthonormal basis for $L^2(X, \mu_c)$.

*Solution.*

Clearly $b(f, \delta_x) = f(x)$ for all $f \in L^2(X, \mu_c)$ and $x \in X$.

Hence $b(\delta_x, \delta_x) = 1$ and $b(\delta_x, \delta_y) = 0$ for all $x, y \in X$ with $y \neq x$.

If $f \perp \delta_x$ for all $x \in X$ then $f(x) = 0$ for all $x \in X$ and so $f = 0$.

**2.** Let $H = \ell_2^\infty$ and let $N$ be the subspace of all sequences $\{x_n\}_{n \geq 1}$ which are ultimately 0, i.e., for which $x_n = 0$ for $n$ sufficiently large.

Show that $N$ is a vector subspace and $N \neq H$, but that $N^\perp = 0$.

*Solution.*

This Hilbert space is the special case of the type in the previous exercise, corresponding to $X = \mathbb{N} = \{1, 2, \ldots\}$.

Let $\delta_n$ be the sequence whose $n^{th}$ term is 1 and with all other terms 0.

As $N$ is the set of all finite linear combinations of the $\delta_n$s it is a vector subspace.

The sequence $\{\frac{1}{n}\}_{n \geq 1}$ is in $H$ but not in $N$, so $N \neq H$.

If $v = \{v_n\}_{n \geq 1} \perp N$ then $v_n = b(v, \delta_n) = 0$ for all $n \geq 1$, and so $v = 0$.

**3.** Let $A : H \to H$ be a linear operator on a Hilbert space $H$ such that

$$||A|| = \sup\{||Ax|| \mid ||x|| \leq 1\} < \infty.$$

Show that $||Ax|| \leq ||A||.||x||$ for all $x \in H$.

*Solution.*

This is clear if $x = 0$. If $x \neq 0$ let $y = \frac{1}{||x||}x$, so $||y|| = 1$.

Then $x = ||x||y$, and $Ax = ||x||Ay$, and so $||Ax|| = ||x||.||Ay|| \leq ||x||.||A||$.

**4.** Let $H$ be a Hilbert space and $N$ the vector subspace generated by a finite orthonormal subset $\{u_1, \ldots, u_n\} \subset H$.

Show that $N$ is closed in $H$.

(The Gram-Schmidt process provides such a basis for any finite-dimensional vector subspace, and so all finite dimensional subspaces are closed.)

*Solution.*

If $\{n_k\}_{k\geq 1}$ is a Cauchy sequence in $N$ then $\beta_{i,k} = b(n_k, u_i)$ is a Cauchy sequrence in $\mathbb{C}$ for each $1 \leq i \leq n$. These sequences converge, with limits $\beta_i$, say. It is easy to see that $\{n_k\}_{k\geq 1}$ converges to $n = \Sigma_{1\leq i\leq n}\beta_i u_i$ in $N$. Hence $N$ is complete and therefore closed.

**5.** Let $\{H_n\}_{n\geq 1}$ be a sequence of Hilbert spaces, with inner products $b_n$ and associated norms $||-||_n$. (Note: here the subscript is merely an index - these are all norms deriving from inner products). Let $H$ be the space of all sequences $\{h_n\}_{n\geq 1}$ with $h_n \in H_n$ for all $n \geq 1$ and $\Sigma_{n\geq 1}||h_n||_n^2 < \infty$.

Show that $H$ is a vector space, and that $b(\{x_n\}, \{y_n\}) = \Sigma_{n\geq 1} b_n(x_n, y_n)$ defines an inner product on $H$ making it into a Hilbert space.

*Solution.*

Argue as the the special case $H_n = \mathbb{C}$ for all $n \geq 1$, which gives $\ell_2^{\infty}(\mathbb{C})$.