Associative Memory Retrieval Induced by Fluctuations in a Pulse Neural Network

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An associative memory retrieval in a pulse neural network composed of the FitzHugh-Nagumo neurons is investigated. The memory is represented in the spatio-temporal firing pattern of the neurons, and the memory retrieval is accomplished using the fluctuation in the system. The storage capacity of the network is investigated numerically. It is demonstrated that this pulse neural network is capable of an alternate retrieval of two patterns.

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I. INTRODUCTION

Recently, there is a considerable attraction of attentions to the associative memory in neural networks composed of model neurons which change their dynamical states temporally, such as, chaotic neurons, oscillator neurons, or spiking neurons [1–14]. They not only arouse the theoretical interests, but also may have a lot to do with the problem of information coding in the brain [15].

Among them, numerous authors investigate the coupled phase oscillators [3–9], which are the general reduced model of the coupled limit-cycle oscillators. All the neurons are oscillating with the almost same period, and the memory is represented in the relative phase differences of oscillators, so they can store the analog-valued patterns. And this model has an advantage that the usual techniques for the theoretical analysis of associative memory [16,17] are applicable.

On the other hand, neural networks composed of spiking neurons also show the properties of associative memory [11–13]. In those systems, following models are often used as spiking neurons, namely, the Hodgkin-Huxley equation which describes the dynamics of squid giant axons, the FitzHugh-Nagumo equation which is the reduced model of the Hodgkin-Huxley equation, or the leaky integrate-and-fire model which has the internal state described by a linear differential equation and a spiking mechanism with a threshold. The couplings among those neurons are accompanied with the time delay which models the time for a pulse to propagate on the axon from the pre-synaptic neuron to the post-synaptic neuron, and the memory is represented in the spatio-temporal firing pattern of the neurons.

Meanwhile, the physiological environment where neurons operate is thought to be highly noisy [18,19], so the effect of the fluctuation may not be neglected. Generally, stochastic resonance (SR) is a well-known phenomenon where a weak input signal is enhanced by its background fluctuation and observed in many nonlinear systems [20–22]. Particularly, SR in a single neuron is well investigated by numerous researchers both experimentally [23,24] and theoretically [25–31], and it is proposed that the biological sensory system may utilize SR to improve the sensitivity to the external input signal. Recently, the effect of SR in spatially extended systems, or neural networks, is investigated, and some new features are reported [32–34]. Concerning SR in the coupled FitzHugh-Nagumo equation, we proposed that the background fluctuation may play a functional role like a parameter of the dynamical system [34].

In the present paper, the associative memory composed of the FitzHugh-Nagumo neurons with the fluctuation is treated, and SR-like effects in this system are considered. In Sec. II, a coupled FitzHugh-Nagumo equation and some quantities are defined. In Sec. III, the results of numerical simulations are presented. The memory retrieval by adding the fluctuation into the system and its dependence on the fluctuation intensity are examined, and an SR-like phenomenon is observed. The basin of the attraction and the storage capacity of the system are also investigated numerically. In Sec. IV, theoretical analyses for the fluctuation-induced memory retrieval are presented. In Sec. V, the simultaneous retrieval of two patterns is observed as the alternate firings of the particular neurons. Conclusions and discussions are given in the last section.

II. ASSOCIATIVE MEMORY COMPOSED OF SPIKING NEURONS

In the following, as a model of associative memory, we treat a coupled FitzHugh-Nagumo (FN) equation written as

$$\tau \dot{u}_i = -v_i + u_i - u_i^3/3 + I_i(t) + \eta_i(t) + \sum_{j=1}^{N} J_{ij}(u_j(t-d_p) - u_{eq}), \tag{1}$$

$$\dot{v}_i = u_i - \beta v_i + \gamma, \tag{2}$$

$$\langle \eta_i(t)\eta_j(t')\rangle = D\delta_{ij}\delta(t-t'), \tag{3}$$

where $\beta = 0.8$, $\gamma = 0.7$, $\tau = 0.1$, $u_{eq} = -1.2$, $d_p = 3$, u_i and v_i denote the internal states of the i-th neuron, $I_i(t)$ is the external input, $\eta_i(t)$ is the Gaussian white noise which represents the fluctuation in the system. Note that a single FN neuron shows the characteristic of the spiking neuron, namely, it has a stable rest state, and with an appropriate amount of disturbance it generates a pulse with a characteristic magnitude of height and width, u_{eq} is the equilibrium value of u_i for $I_i(t) = 0$, $\eta_i(t) = 0$, and $J_{ij} = 0$ $(i, j = 1, 2, \cdots, N)$, and that d_p is the uniform propagational time delay.

Then let us make the above N neurons store p random patterns ξ_i^μ $(i = 1, 2, \cdots, N,\ \ \mu = 1, 2, \cdots, p)$, generated according to the probability density function

$$P(\xi_i^\mu) = (1-a)\delta(\xi_i^\mu) + a\delta(\xi_i^\mu - 1), \tag{4}$$

where $\delta(x)$ denotes the delta function and a $(0 \leq a \leq 1)$ is the average of ξ_i^μ. Following Yoshioka and Shiino [13], the connection coefficients J_{ij} are defined as

$$J_{ij} = \frac{w}{Na(1-a)} \sum_{\mu=1}^{p} \xi_i^\mu(\xi_j^\mu - a), \tag{5}$$

where the parameter w scales the strength of J_{ij} and is fixed at $w = 0.15$ in the following. Note that the matrix $J_{ij} \propto \sum_\mu \xi_i^\mu(\xi_j^\mu - a)$ is used instead of usual $J_{ij} \propto \sum_\mu(\xi_i^\mu - a)(\xi_j^\mu - a)$ so as not to give the negative input to the neurons which store 0's, because the FN neuron can fire even with the negative input due to the rebound effect [35].

The external input $I_i(t)$ is defined as

$$I_i(t) = Ix_i\Theta(t) \qquad (x_i \in \{0, 1\}), \tag{6}$$

where I is the strength of the external input, x_i is the binary factor which determines whether the input is injected to the i-th neuron or not, and $\Theta(t)$ is Heaviside's step function which takes 1 for $t \geq 0$ and otherwise takes 0. In the following, I is fixed at $I = 0.1$, which is so small that each neuron can not fire without the fluctuation $\eta_i(t)$. Using the binary factor x_i, the input-overlap m_{in}^μ, which measures the correlation between the pattern $\xi^\mu = (\xi_1^\mu, \xi_2^\mu, \cdots, \xi_N^\mu)$ and the external input $I(t) = (I_1(t), I_2(t), \cdots, I_N(t))$, is defined as

$$m_{in}^\mu = \frac{1}{Na(1-a)} \sum_{i=1}^{N} (\xi_i^\mu - a)(x_i - a). \tag{7}$$

III. FLUCTUATION-INDUCED MEMORY RETRIEVAL

Following the above configurations, numerical simulations are carried out for $N = 200$, $p = 3$, and $a = 0.5$. Without loss of generality, the pattern ξ^1 can be defined as

$$\xi_i^1 = \begin{cases} 1 & 1 \leq i \leq 100 \\ 0 & \text{otherwise} \end{cases}, \tag{8}$$

and the pattern ξ^2 and ξ^3 are determined randomly following the probability density function (4). The external input is derived by determining the binary factors x_i randomly so that the input-overlap m_{in}^1 with the pattern ξ^1 takes 0.5. A typical time series of $u_1(t)$ for the fluctuation intensity $D = 0.001$ is shown in Fig. 1, where the fluctuation around u_{eq} and the two firings are observed.

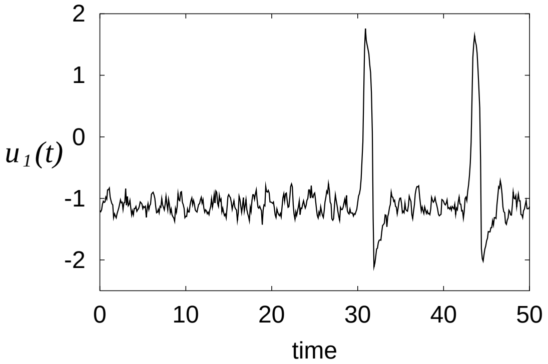

FIG. 1. A typical time series of $u_1(t)$ for $N = 200$, $p = 3$, $a = 0.5$, and $D = 0.001$. The fluctuation around u_{eq} and the two firings are observed.

To measure the correlation between the pattern ξ^μ and the time series $u_i(t)$ $(i = 1, 2, \cdots, N)$, $u_i(t)$ is transformed into the binary series $y_i(t) \in \{0, 1\}$. Firstly, let us define the firing time of the i-th neuron as the time when $u_i(t)$ exceeds an arbitrary threshold θ, and we set $\theta = 0$ in the following. Then the time series $u_i(t)$ is transformed into the binary series

$$y_i(t) = \begin{cases} 1 & t < t_i^f + d \\ 0 & \text{otherwise} \end{cases}, \tag{9}$$

where t_i^f is the latest firing time of i-th neuron at time t, and the parameter d is set close to the characteristic width of the output pulse and $d = 4$ is used in the following. Then the output-overlap m_{out}^μ between the pattern ξ^μ and the binary series $y = (y_1(t), y_2(t), \cdots, y_N(t))$ is defined as

$$m_{out}^\mu = \frac{1}{Na(1-a)} \sum_{i=1}^{N} (\xi_i^\mu - a)(y_i - a). \tag{10}$$

The firing times of all the neurons for the fluctuation intensity $D = 0.001$ are shown in Fig. 2 (a), and it is observed that all the neurons are firing randomly.

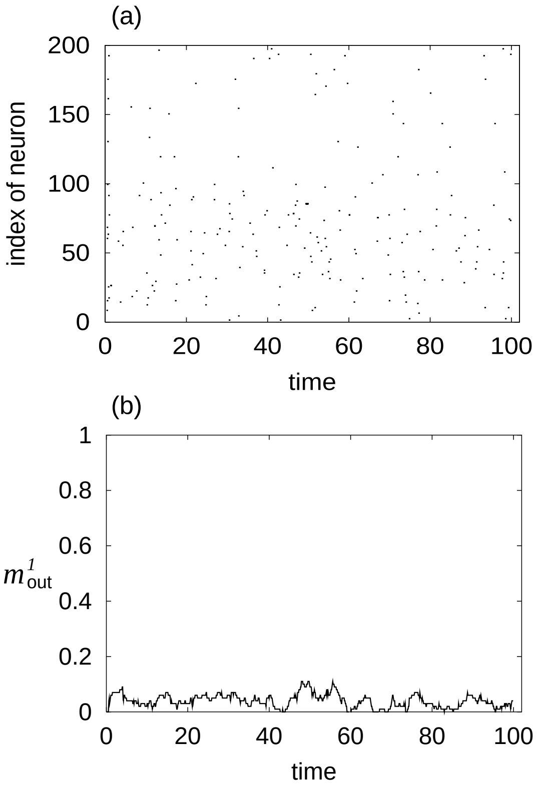

FIG. 2. The result of numerical simulation, (a) the firing times of all the neurons and (b) the output-overlap m^1_{out} with the pattern ξ^1, for $N = 200$, $p = 3$, $a = 0.5$, and $D = 0.001$. All the neurons are firing randomly, so the retrieval of the pattern ξ^1 fails.

The output-overlap m^1_{out} with the pattern ξ^1 obtained from the time series in Fig. 2 (a) is shown in Fig. 2 (b). It is observed that m^1_{out} fluctuates around 0, so it can be concluded that the retrieval of the pattern ξ^1 fails.

The firing times of all the neurons for $D = 0.002$ are shown in Fig. 3 (a).

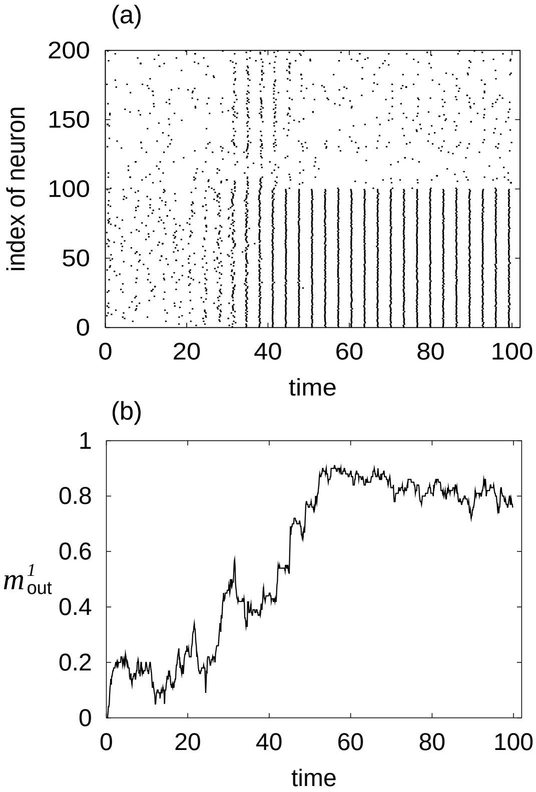

FIG. 3. The result of numerical simulation, (a) the firing times of all the neurons and (b) the output-overlap m^1_{out} with the pattern ξ^1, for $N = 200$, $p = 3$, $a = 0.5$, and $D = 0.002$. The retrieval of the pattern ξ^1 is successful.

It is observed that all the neurons seem to fire randomly at small t, but at $t \sim 40$, the neurons which store 1's for the pattern ξ^1 start to fire periodically and synchronously. And in Fig. 3 (b), the output-overlap m^1_{out} increases to about 0.8 at $t \sim 40$, so in this case the retrieval of the pattern ξ^1 is successful.

The results of the simulation for $D = 0.004$ are shown in Fig. 4.

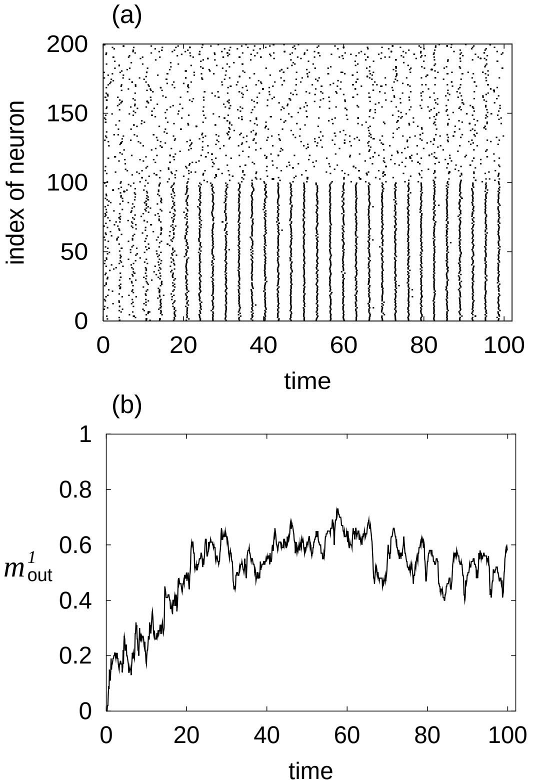

FIG. 4. The result of numerical simulation, (a) the firing times of all the neurons and (b) the output-overlap m^1_{out} with the pattern ξ^1, for $N = 200$, $p = 3$, $a = 0.5$, and $D = 0.004$. The neurons which store 0's for pattern ξ^1 fire with high firing rates due to the large fluctuation intensity, so the output-overlap is lower than the case of $D = 0.002$.

The periodic and synchronous firings are observed again, but the neurons which store 0's for pattern ξ^1 also fire with high firing rates due to the large fluctuation intensity, so the output-overlap is lower than the case of $D = 0.002$.

In Fig. 5, the output-overlap m^1_{out} at a sufficient large t is plotted against the fluctuation intensity D for the input-overlap $m^1_{in} = 0.8$, 0.6, and 0.1.

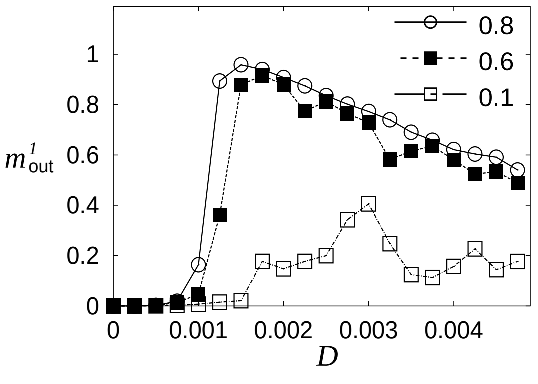

FIG. 5. The output-overlap m^1_{out} against the fluctuation intensity D for $m^1_{in} = 0.8, 0.6$, and 0.1 with $N = 200$, $p = 3$, and $a = 0.5$. Stochastic resonance-like phenomenon is observed for $m^1_{in} = 0.8$ and 0.6.

The other parameters are identical with the previous cases. For $m^1_{in} = 0.8$ and 0.6, the output-overlap m^1_{out} increases with the increase of the fluctuation intensity D, and it decreases with the increase of D over the optimal intensity $D_0 \sim 0.0015$. This phenomenon is similar to so-called stochastic resonance, where a weak input signal is enhanced by its background fluctuation and observed in many nonlinear systems [20–22]. For $m^1_{in} = 0.1$, the retrieval of pattern ξ^1 fails for any value of D.

For the fixed fluctuation intensity $D = 0.002$, the numerically obtained basin of attraction is shown as a function of the loading rate $\alpha = p/N$ in Fig. 6.

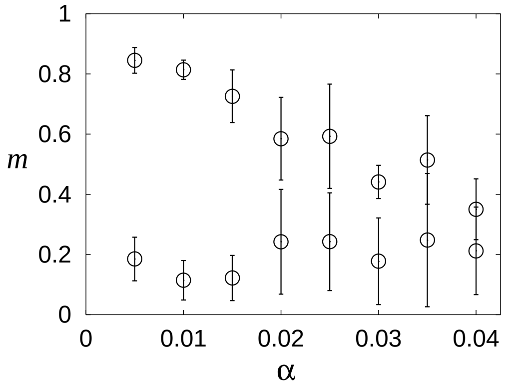

FIG. 6. The basin of attraction for $N = 200$, $D = 0.002$, and $a = 0.5$. The error bar denotes the standard deviation for ten samples. The storage capacity is estimated to be about 0.02.

For each loading rate α, two points are plotted, namely, the upper is the equilibrium value of the output-overlap m^1_{out}, and the lower is the minimum input-overlap m^1_{in} which gives the successful memory-retrieval. For $\alpha < 0.02$, the standard deviations shown by the error bars are relatively small, but for $\alpha \geq 0.02$, they take larger values, that is, the memory-retrieval states are destabilized. So it can be concluded that the storage capacity α_c is about 0.02. For further discussions, theoretical analyses of the associative memory [16,17] are needed.

IV. THEORETICAL ANALYSIS OF FLUCTUATION-INDUCED MEMORY RETRIEVAL

A. Fluctuation-induced memory retrieval

In this section, we give the qualitative explanation for the fluctuation-induced memory retrieval. In the following, the system with $p = 1$ is considered for simplicity. Let us define the set of indices of neurons which store 0's in the pattern $\xi^\mu = (\xi^\mu_1, \xi^\mu_2, \cdots, \xi^\mu_N)$ as $G^\mu(0)$, and the set of indices of neurons which store 1's in the pattern ξ^μ as $G^\mu(1)$. The input K_i injected to the i-th neuron is written as

$$K_i = \eta_i \quad \text{for } \ i \in G^1(0), \tag{11}$$

$$K_i = \frac{w}{Na(1-a)} \sum_{j=1}^{N} (\xi^1_j - a)(u_j - u_{eq}) + I + \eta_i, \tag{12}$$

$$= \frac{w}{Na(1-a)} \left(\sum_{j \in G^1(0)} (\xi^1_j - a)(u_j - u_{eq}) + \sum_{j \in G^1(1)} (\xi^1_j - a)(u_j - u_{eq}) \right) + I + \eta_i, \tag{13}$$

$$= w \left(-\frac{1}{N(1-a)} \sum_{j \in G^1(0)} (u_j - u_{eq}) + \frac{1}{Na} \sum_{j \in G^1(1)} (u_j - u_{eq}) \right) + I + \eta_i, \tag{14}$$

$$= w \left(-\langle u_j - u_{eq} \rangle_{j \in G^1(0)} + \langle u_j - u_{eq} \rangle_{j \in G^1(1)} \right) + I + \eta_i \quad \text{for } \ i \in G^1(1), \tag{15}$$

where $\langle \cdot \rangle_{j \in A}$ denotes the ensemble average over the set A. Note that the external input $I(t)$ is injected only to the neurons in $G^1(1)$ for simplicity. Because noises for different neurons are statistically independent, the neurons in $G^1(0)$ fire randomly and independently. On the other hand, the neurons in $G^1(1)$ have the common input $w\langle u_j - u_{eq} \rangle_{j \in G^1(1)}$, so their firings may be correlative each other. In the following, we treat this dynamics.

Let us consider an ensemble of N neurons with the uniform coupling term $w\langle u_j(t - d_p) - u_{eq} \rangle_j$ and the external input $I + \eta_i$, namely, Eqs. (1) and (2) with $J_{ij} = w/N$ and $I_i(t) = I$. Note that this model approximates the dynamics of neurons in $G^1(1)$, and that the term $\langle u_j - u_{eq} \rangle_{j \in G^1(0)}$ in Eq. (15) is neglected for simplicity. Then let us consider the number of neurons which fire in the narrow time interval $[t, t + \Delta]$ and denote it by Nz_n. If an output pulse of FN neuron has width d and height M, the perturbation with width $\sim d$ and height $\sim wMz_n$ is injected to all the neurons with the delay d_p. Let us denote the number of neurons which fire with this perturbation in the time interval $[t + d_p, t + d_p + \Delta]$ by Nz_{n+1}, and assume the relation $z_{n+1} = g(z_n)$. If the FN neuron acts like a threshold device with the threshold I_0, $g(z_n)$ for noise intensity $D = 0$ is a step function which takes 1 for $wMz_n + I \geq I_0$ and takes 0 otherwise. It is difficult to derive $g(z_n)$ for $D \neq 0$, but it is expected to be a monotonic increase function of z_n.

Numerically obtained $g(z_n)$ for $D = 0.0005$, 0.001, and 0.0012 with $N = 100$ and $I = 0.1$ is plotted in Fig. 7.

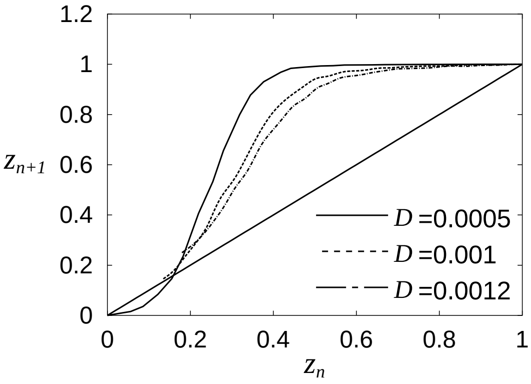

FIG. 7. Numerically obtained $g(z_n)$ for D=0.0005, 0.001, and 0.0012 with $N = 100$ and $I = 0.1$. A saddle-node bifurcation at $D \sim 0.001$ is observed.

The width Δ of time interval is set at the same size with d of output pulse. It is observed that the number of intersecting points of $y = g(z)$ with $y = z$ is 3 for $D < D_0 \sim 0.001$, and 1 for $D > D_0$, the intersecting point $z \sim 1$ is always stable for any D, and that the other intersecting points are generated by a saddle-node bifurcation at $D = D_0$. The schematic diagram is shown in Fig. 8.

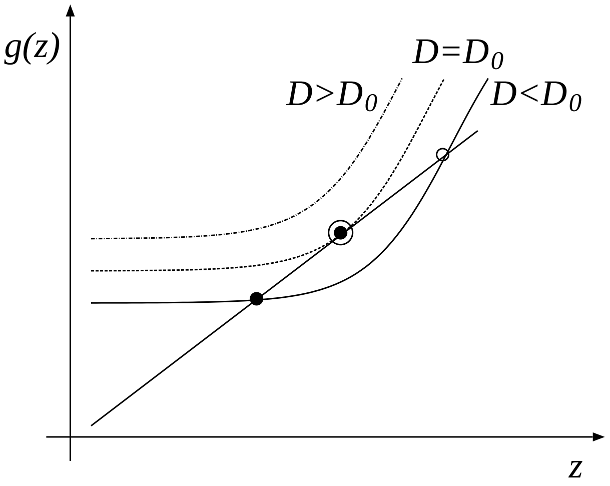

FIG. 8. Schematic diagram of bifurcation of $g(z)$.

Thus, for $D > D_0$, any z_n converges to the stable fixed point $z_n \sim 1$, which means that all the neurons fire synchronously and periodically with the period d_p for $D > D_0$.

B. The dependence of m_{out} on D

In this section, the dependence of m_{out} on D is investigated for $D > D_0$. Assume that the neurons in $G^1(1)$ fire synchronously and periodically with the period d_p and that the neurons in $G^1(0)$ are firing randomly with firing rate depending on D as $r_{G0} = r_0 \exp(-C/D)$, where r_0 and C are constants. Note that this firing rate is the inverse of the first passage time for a particle in a double well potential to cross the potential barrier [36], and introduced only for simplicity.

The distribution of the ratio k of the neurons in $G^0(0)$ which fire in a time interval of width d, and its average $\langle k \rangle$ are written as

$$
\begin{aligned}
P(k) &= {}_{N(1-a)}C_{N(1-a)k}(1-\exp(-r_{G0}d))^{N(1-a)k}(\exp(-r_{G0}d))^{N(1-a)(1-k)}, &(16)\\
\langle k\rangle &= 1-\exp(-r_{G0}d). &(17)
\end{aligned}
$$

With $\langle k\rangle$, m_{out} is approximately given by

$$
\begin{aligned}
m^1_{out} &= \frac{1}{Na(1-a)}\sum_i(\xi^1_i-a)(y_i-a), &(18)\\
&= \frac{1}{Na(1-a)}((1-a)(1-a)Na+(-a)(1-a)N(1-a)\langle k\rangle\\
&\quad +(-a)(-a)N(1-a)(1-\langle k\rangle)), &(19)\\
&= \exp\left(-r_0 d\exp\left(-\frac{C}{D}\right)\right). &(20)
\end{aligned}
$$

Note that Eq. (20) decreases monotonically with the increase of D. This gives the quantitative description of the decrease of m_{out} for $D \geq D_0$.

V. ALTERNATE RETRIEVAL OF TWO PATTERNS

In our network, the memory is represented by the synchronized periodic firings of the neurons which store 1's, and this period is determined by the propagational time delay d_p. So the system has a large degree of freedom along the time axis for the large d_p, that is, during the time between the firings by one pattern, the system can retrieve other patterns, in other words, this system can process some "tasks" simultaneously.

To see this ability, numerical simulations are performed for $N = 200$, $p = 3$, $a = 0.5$, and $d_p = 6.5$. Note that the propagational time delay d_p is about twice as long as $d_p = 3$ used in above sections. For simplicity, the pattern ξ^1 and ξ^2 are defined as

$$
\xi^1_i = \begin{cases} 1 & 1 \leq i \leq 100 \\ 0 & \text{otherwise} \end{cases}, \qquad (21)
$$

$$
\xi^2_i = \begin{cases} 1 & 51 \leq i \leq 150 \\ 0 & \text{otherwise} \end{cases}, \qquad (22)
$$

respectively, and the pattern ξ^3 is determined randomly following the probability density function (4). The external input $I(t)$ is defined so that the binary factor x_i suffices

$$
x_i = \begin{cases} 1 & 51 \leq i \leq 100 \\ 0 & \text{otherwise} \end{cases}. \qquad (23)
$$

Note that both input-overlaps m^1_{out} and m^2_{out} take 0.5.

For the fluctuation intensity $D = 0.001$, the firing times of all the neurons and the output-overlaps m^1_{out} and m^2_{out} are plotted in Fig. 9 (a) and Fig. 9 (b) respectively.

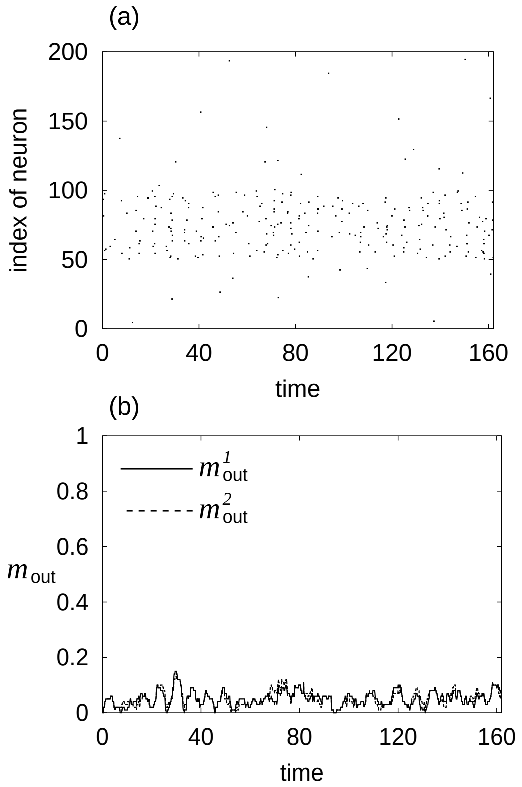

FIG. 9. The results of numerical simulation, (a) the firing times and (b) the output-overlaps, for $N = 200$, $p = 3$, $a = 0.5$, $D = 0.001$, and $d_p = 6.5$. The retrievals of both pattern ξ^1 and pattern ξ^2 fail.

It is observed that the retrievals of both pattern ξ^1 and pattern ξ^2 fail with this fluctuation intensity. The firing times of all the neurons for the fluctuation intensity $D = 0.002$ are plotted in Fig. 10 (a).

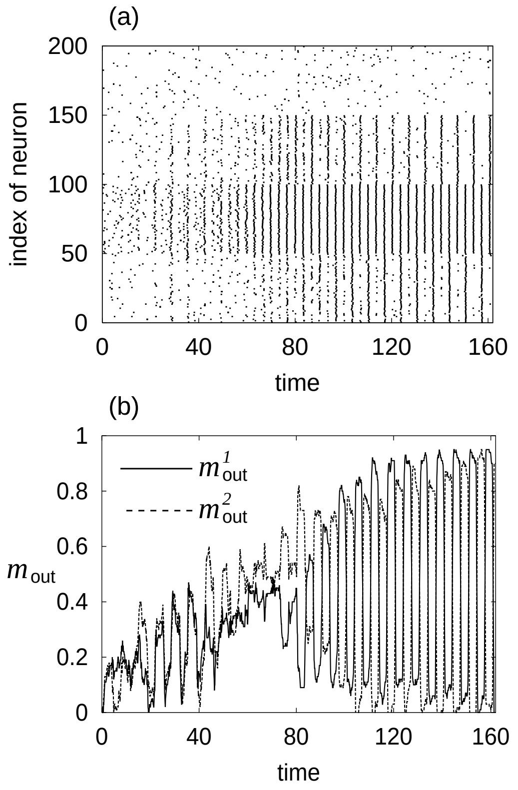

FIG. 10. The results of numerical simulation, (a) the firing times and (b) the output-overlaps, for $N = 200$, $p = 3$, $a = 0.5$, $D = 0.002$, and $d_p = 6.5$. The alternate retrieval of two patterns is observed as the anti-phase oscillations of two output-overlaps.

It is shown that the two patterns ξ^1 and ξ^2 are retrieved alternatively, accompanied with the time difference $d_p/2$. The output-overlaps m^1_{out} and m^2_{out} derived from the data in Fig. 10 (a) are shown in Fig. 10 (b). The alternate retrieval of two patterns is observed as the anti-phase oscillations of two output-overlaps.

The results of the numerical simulation for $D = 0.004$ are shown in Fig. 11.

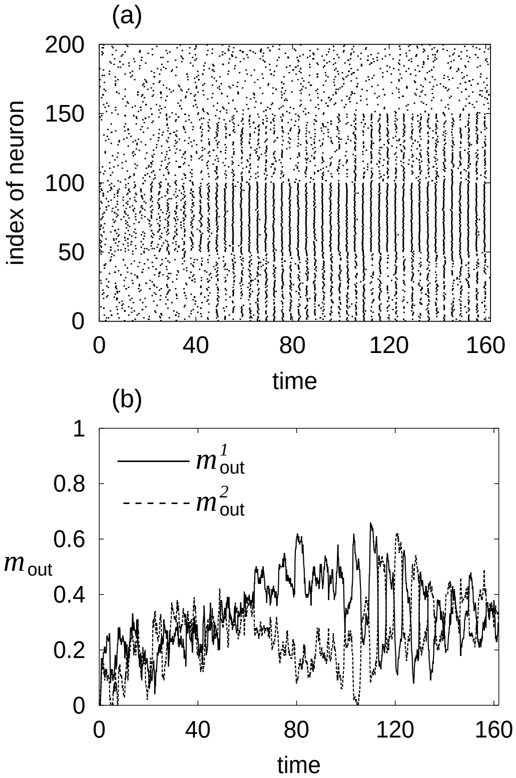

FIG. 11. The results of numerical simulation, (a) the firing times and (b) the output-overlaps, for $N = 200$, $p = 3$, $a = 0.5$, $D = 0.004$, and $d_p = 6.5$. All the neurons are firing with high firing rates, so the retrievals of both pattern ξ^1 and pattern ξ^2 fail.

In Fig. 11 (a), it is observed that all the neurons are firing with high firing rates, so the retrievals of both pattern ξ^1 and pattern ξ^2 fail as in Fig. 11 (b).

From above results, it can be concluded that our system has an ability to retrieve two patterns simultaneously as the alternate firings of particular neurons, and the fluctuation intensity D plays a significant role to realize this dynamics.

VI. CONCLUSIONS AND DISCUSSIONS

The associative memory in a pulse neural network composed of the FitzHugh-Nagumo neurons with the propagational time delay is investigated. In this network, the memory is represented by the synchronous periodic firings of the particular neurons. It is found that the memory retrieval in this system is achieved by adding the fluctuation, and there exists an optimal fluctuation intensity for the memory retrieval. This phenomenon is similar to so-called stochastic resonance (SR), where the weak input signal is enhanced by its background fluctuation. Though there is no time-dependent input in our model, the mechanism of associative memory is driven and enhanced by its background fluctuation. The basin of attraction of this system is investigated numerically, and its storage capacity is found to be $\alpha_c \sim 0.02$. Note that this storage capacity is smaller than those of previous models, for example, 0.138 for the Hopfield model [37], and 0.038 for the coupled phase oscillators [38]. But our network has an ability that the previous models do not have, that is, an ability to retrieve two patterns as the alternate firings of the particular neurons. While such dynamics as utilizes the degree of freedom along the time axis is proposed by Wang et al. for the network of bursting neurons [14], our model has the properties that the component of the memory is the single pulse of each neuron, and that the fluctuation in the system is indispensable.

As for the fluctuation in the neural system, SR in a single neuron is often investigated, and it is proposed that the sensory system may utilize SR in order to improve the sensitivity to the external input. Our results show that the fluctuation can play more functional role in higher order dynamics in the brain, like the memory retrieval in the associative memory. Though Collins et al. propose that the regulating of the fluctuation intensity is not required for the network of large number of neurons [33], but in our dynamics, it is required to regulate the fluctuation intensity to the optimal intensity (see Fig. 5). It might be difficult to regulate the fluctuation intensity if the fluctuation in our model is considered to be the thermal noise in the neural system, but that might be naturally performed if the fluctuation in our system represents the sum of enormous pulses from the pre-synaptic neurons [29–31]. In such case, the dynamics of the system might be controlled by its background fluctuation [34].

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