A Small World without Random Graphs

Robert E. Hiromoto
Department of Computer Science
University of Idaho
Moscow, Idaho 83844-1010
USA
hiromoto@uidaho.edu

***Abstract*— Small-world networks provide a promising depiction of social, communication, and biological behavior as found in practice. These networks are subset of random graphs where edges are randomly re-wiring or added onto a regular lattice. The small-world phenomena display high, localized clustering and short paths between vertices. These properties are seen to exist in the real-world. The small-world models apply randomness to design complex systems with these properties. This paper takes a more deterministic approach to small-world networks. The paper presents a neuronal-axon network architecture that takes into consideration the network complexity in the number of links or edges out of each node, and the memory capacity of the individual nodes within the network. The resulting network architecture exhibits the small-world clustering with a logarithmic degree of separation between nodes without the need for long-range communication edges.**

***KeyWords*—Random Graphs, Small-World Networks, Deterministic, Neuronal Networks**

I. Introduction

A small-world network is the notion that almost any pair of people in the world can be connected to one another by a short chain of intermediate acquaintances, characterized by a separation length of about six. Small-world models for social networks should display a large clustering coefficient; a high local clustering of disjoint regions that on average are connected to any node by only a few steps. A short summary of some background and models attempting to explain small-world phenomena are presented in [2; 21; 24; 27].

This paper presents an overview of small-world models, based on rules applied with small random probabilities. This approach makes sense if Nature is truly stochastic [7]. As a modeling tool, a random approach provides a high-level characterization of a complex behavior with minimal details of physical or systemic rules. However, it can also be argued that without specific details of resource limitations, costs, the advance in the prevailing technologies or the economics of scale; the insight afforded by random approaches may be obscured. This paper presents a different perspective by examining the design of a neuronal network simulator that is found to exhibits small-world features. In the remainder of this paper, we present the notion of six degrees of separation, current small-world models, the design of a neuronal-axon network simulator, its applicability to a small-world neuronal network, and an argument this small-world network based on experimental evidence.

A. Six Degrees of Separation

Milgram [20] provided one of the first studies in quantifying the small-world property of networks. Since then, the notion of "six degrees of separation" [11] was introduced. His work explored the average number of steps that separated two individual from each other through a social network of friends-of-friends. The average separation is the number of steps that a message is passed between individuals, chosen from a random source, to arrive at a particular destination. The results of the experiments shows that the average number of steps needed is surprisingly small.

The small-world, social network relies on two properties: 1) on average a person's friend-of-a-friend are far more likely to be acquainted with one another than two people chosen at random (clustering); and, 2) it is possible to connect two people chosen at random via a chain of only a few intermediate acquaintances (six degrees of separation).

A slightly different approach, is to construct short paths based on local information. When network models are able to do this efficiently, the network is called *navigable*. Local information is defined as messages passed only between friends without knowing the exact path on subsequent passes. One strategy is to pass the message to the friend that appears closest to the destination (as measured in a social relational sense). Algorithmically, this approach implies a variant of a *greedy routing strategy* to find short paths in a network. This strategy forms the basis for a decentralized view that is used for navigation and searching in small-world models.

Small-world networks have importance in modeling the spread of diseases [19; 23], the design and analysis of the internet infrastructure [1], and the neural-axon clustering networks of the brain, to mention only a few application areas.

B. Random Graphs

The theory of random graphs [3; 8] provides a framework where short paths can exist in large networks. Such a graph is denoted as $G(N, p)$, where N is the number of nodes (vertices) and p is a fixed probability of connecting an edge between a pair of vertices. Assume that z number of edges per node (on average). Then the number of edges between nodes in the graph is $\frac{1}{2}Nz$. Now select N nodes and draw $\frac{1}{2}Nz$ edges between randomly selected pairs. Each pair of

nodes are connect with an edge with probability p. An graph with no edges has $p = 0$; whereas, a fully connected graph has $p = 1$.

To understand the small-world effects of random graphs, consider a node N_i with z neighbors. Assume that each of N_i's neighbors also has z neighbors, which implies that N_i with z^2 second neighbors. By extending this argument, the number D degrees of separation needed to reach all N nodes in the network is given by

$$z^D = N \rightarrow D = \frac{logN}{logz} \tag{1}$$

It is possible that the second neighbors of N_i are also neighbors of N_i. In that case, these nodes form a clustering of the network. Unfortunately, the clustering of networks in a random graph is diminished by the likelihood that p selects pairs of nodes to connect in a uniformly random way.

The clustering coefficient C = average fraction of pairs of neighbors of a node that are also neighbors of each other. A fully connected network has a clustering coefficient $C = 1$. The clustering coefficient for a random graph is $C = \frac{z}{N}$.

A measure for typical separation between vertices in a graph is denoted by $L(p)$ where p measures the range of randomness for a graph, $0(ordered) < p < 1(disordered)$.

In Table 1, three different networks are analyzed by Watts and Strogatz [28]. In the table, the corresponding z values (not shown) are $z = 2.67$ for the power grid (western USA); $z = 14$ for the C. elegans; and $z = 61$ for the movie actors.

Network	N	L_{actual}	L_{rand}	C_{actual}	C_{rand}
C. elegans	282	2.65	2.25	0.28	0.05
Power Grid	4941	18.7	12.4	0.08	0.005
Film actor	225,226	3.65	2.99	0.79	0.00027

Table 1. Watts and Strogatz (1998)

II. Building Graphs with both Small-World and Clustering Properties

Random graphs show small-world properties but restrict the formation of clusters of vertices. Since clustering is an important property of real-world situations, a graphical network model has significance in the analysis and modeling of both physical and social networks [17]. In this section we will consider the models of Watts and Strogatz, and Kleinberg.

A. Watts-Strogatz Model

The model of Watts and Strogatz [28] is based on the assumption that people are more likely to have friends nearby, but still have some friends that live at a far distance. With these assumptions, the Watts-Strogatz's small-world model exhibits the high clustering found in social networks.

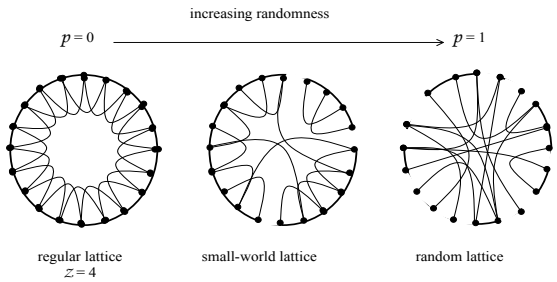

Fig. 1. Lattices: Ordered, Small-World, Random

The Watts-Strogatz graph is a one-dimensional, regular circular lattice (*discrete, and with periodic boundary conditions*) that allows a small degree of randomness to produce the small-world effect (see Figure 1). Initially each node is assigned a position in the lattice and an edge is added between each node and its nearest neighbors to form the one-dimensional lattice. Next a fixed (but small) number of edges are randomly rewired between nodes and may stretch long distances within the lattice.

The parameter, p, denotes the fraction of edges that are subject to random rewiring. The variation of p makes possible the transition from a very locally ordered/clustered graph ($p = 0$) to one where all edges are randomly rewired ($p = 1$). The rewired edges create *shortcuts* that cover large distances in the underlying lattice. These shortcuts make possible the access, within a small number of steps, vertices that are far away in the graph.

The Watts-Strogatz model exhibits a rapid drop in the shortest path length and network diameter with only a small fraction of random rewiring. Albert and Barabási [2] provide additional details and simulations.

B. Kleinberg Model

Kleinberg (2000) [16] generalizes the Watts-Strogatz model and argues that models with shortcuts that are arbitrarily far apart are poor representations for some real-world situations. Kleinberg observations that people using local information are able to find short paths between individuals. He shows that the Watts-Strogatz model does not allow an efficient greedy routing algorithm to find short paths given only local information.

Kleinberg's model is a k-dimensional lattice with local connections to the nearest-neighbors. The probability for a shortcut edge to be added between two nodes is proportional to a distance measure $d(x, y)$ between each pair of nodes x and y that falls off as a power law proportional to $d(x, y)^{-p}$. Kleinberg shows that when $p = k$ an efficient greedy routing algorithm can be constructed to find short paths with only the use of local information. The corresponding probability for adding a shortcut edge from x to y is given by

$$p(x \leftrightarrow y) = \frac{d(x,y)^{-k}}{H_k(n)} \tag{2}$$

where $H_k(n)$ is a normalization constant.

The Kleinberg model adds shortcuts between nodes; whereas, the Watts-Strogatz model rewires existing edges in the lattice. For $p = 0$, the Kleinberg model corresponds roughly to the Watts-Strogatz model. As p increases, the length of a shortcut edge becomes shorter with respect to the distance between nodes in the lattice. k is the critical value for p that couples the dimensionality of the lattice with the probability distribution needed to balance the shortcut distances in this greedy routing algorithm. An algorithm where shortcuts have been generated under this distribution takes an average complexity of $O(log^2(n))$ steps.

Newman and Watts [22] also considered the idea of adding shortcut edges in place of rewiring existing edges.

III. A Large-Scale Biophysical Neuronal-Axon Network

In this section, a parallel neuronal-axon network simulator is described that is designed to maximize the memory utilization of the controlling processor(s) while at the same time minimizes the edges (communication connections) required to perform efficient global data transfers. The resulting implementation is shown to have small-world and clustering properties.

The Hodgkin-Huxley (H-H) equations [13] models the currents through the membrane of the squid giant axon. Modeled as a long cylindrical tube, electrical signals propagate along outer membrane of the axon. The axon membrane is permeable to preferential chemical elements; namely potassium (K+) and sodium (Na+). Transmembrane potential difference (rest state) $\approx$ -70 mV. The H-H equations describe the sum of ion channels (K+ and Na+) and the capacitance of the axon membrane.

The parallel neuronal network simulator simulates the synaptic response to the Hodgkin-Huxley model of the axon. Each (neuron) processors is given an equal portion of the total neurons to process to ensure load balancing. Neuron cells communicate through a stimulus (action potential). When a neuron cell is stimulated above a given threshold, it passes on that stimulus to cells that it is connected. In a typical model simulator, the neuron cells are connected to each other. To make communication possible between neuron cells that are not in the same processor, neuron processors require access to all other processors. In the worst case, for P neuron processors there can be $O(P^2)$ communication links that communicate at the same time t. Given the large number of neuronal cells (approx. 10,000) that one neuron can stimulate, the neuronal connection must either be prohibitively large (e.g., $O(N^2)$ or employ a small-world degrees of separation. Suppose that the aggregate of activation potential A_N from all N contributing neurons arriving at a single neuron at one instant in time is given by

$$A_N = \Theta(\sum_{i=1}^{N} \triangle_i) \tag{3}$$

where $\triangle_i$ is the individual action potential sent by the i^{th} neuron.

If

$$\triangle_i \propto \frac{A_{max}}{M}, \tag{4}$$

it is then possible that when $M < N$,

$$A_N > A_{max} \tag{5}$$

resulting in a neuron input saturation. One wonders then whether a small-world network for a neuronal-axon network has deeper physical implications.

The parallel simulator would also benefit from small-world network properties. With this in mind, one rather simple parallel network topology is the Master/Slave architecture as illustrated in Figure 2. This communication topology is simple to configure, it has minimal distance between the Master node to each of its slave nodes, and can be built from off-the-shelf components.

In the Master/Slave topology, the number of network connections grow linearly in the number of processors (P). From a pedestrian's point of view, this topology is compact; appears to utilize the processors efficiently; and requires minimal communication hardware (i.e., channels). Thus an elegant solution. However, the reality is that serious tradeoff lurk in this design. First, most parallel processing systems cannot perform I/O in parallel on a single processing node, thus the Master node is hopelessly sequentialized as P increases. Second, the *memory* capacity for the activation potentials (tokens) that may number upwards of 10,000 will eventually surpass the available memory allocations of the Master node that handles the distribution of tokens to the appropriate nodes.

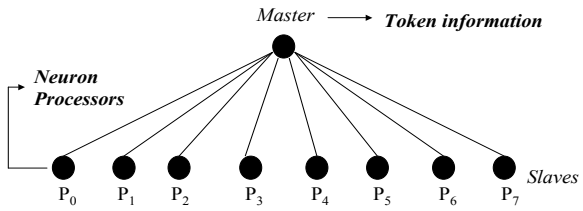

Fig. 2. Master/Slave Topology

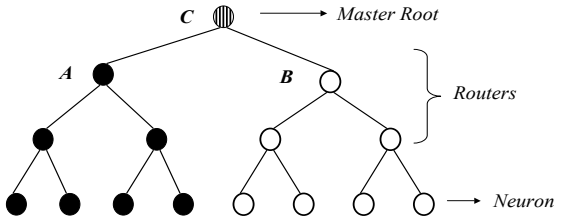

Fig. 3. Binary Tree Topology

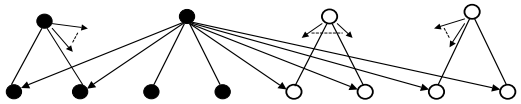

Fig. 5. ThirdLevelBisection

A second architectural approach considered, requires the use of additional processors; however, it also has the advantage of scaling linearly in the number of connections as P increases. This topology is a tree. In the rest of this discussion, it is assumed that the tree topology is a balanced binary tree. Figure 3 illustrates the tree structured neuronal simulator. The tree-structured simulator is organized into three distinct regions. At the leaf-nodes, P_i are the neuron processors. The nodes in the interior of the tree are routing nodes that sort activation tokens that they receive and pass these tokens either up towards the root node or down if the tokens are destined to any child node within their sub-tree. One essential advantage of the strategy is that not all activation tokens needs to reach the root node as was the case for the Master/Slave topology. The root node is now responsible for routing tokens either to its right sub-tree or to its left sub-tree. Another advantage derived from the tree-structure is the incremental sorting of activation tokens that are asynchronous distributed to the respective destination nodes. This strategy results in processing larger number of neurons that could not be performed using the Master/Slave network topology.

Eventually, the tree-structured simulator will face memory problems as well. The solution to this problem takes advantage of the tasks performed by the routers and the Master node. Specifically, all tokens that arrive at the Master node are destined for its right or left subtree but this fact is also known by the corresponding child nodes that are one level below the Master node. The solution illustrated in Figure 4 removes the Master node (i.e., bisect the binary tree) and add shortcut edges (indicated by the dashed arrows) to the two routing nodes labeled A and B. The solution does increase the connection complexity but at the same time increases the clustering coefficient for *specific* nodes in a non-random) way.

The tree bisection with added shortcuts is not limited to the root level alone. In fact we have performed experiments where the bisection went to the extreme level of $log_2(P)-1$ (see Figure 5 and still achieved additional speedup.

The approach taken to maintain efficient memory and routing for the neuronal-axon simulator has also introduced long distance shortcuts. Recall that the Watts-Strogatz model also suffers from the introduction of long distance shortcuts. Kleinberg's model avoided long distance shortcuts by the use of a power law distribution. Although the latter approach applies in both low- and high-dimensional spaces, we avoid long distance shortcuts by introducing a high-dimensional hierarchical organization that accomplishes the same goal (this is not to say that Kleinberg's approach could not be incorporated in to our approach as well). Tanay et al., [25] give evidence of such a hierarchical organization in the yeast molecular network. Figure 6 illustrates how the added dimensionality can re-scale the distance measure between points.

The bisected tree graph with shortcuts can be reconfigured in a three-dimensional space by first applying a two-dimensional rotation on one of the sub-trees (see Figure 7) followed by an overlay in the third dimension. Figure 8 illustrates the final network topology that is free of long distance shortcuts.

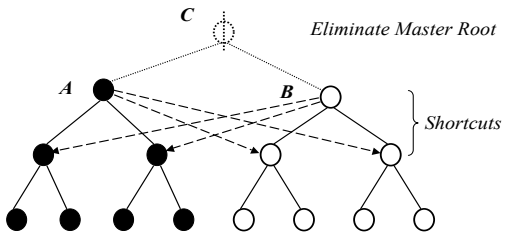

Fig. 4. Bisected-Binary Tree Topology

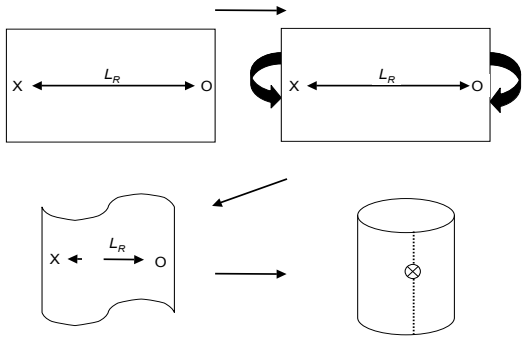

Fig. 6. 3-Dimensional Space

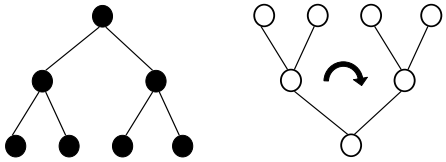

Fig. 7. Two-Dimensional Rotation

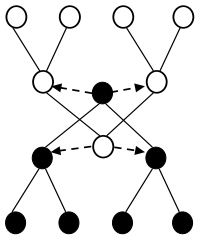

Fig. 8. Overlay in Third Dimension

IV. A Small-World for a Neuroanatomy model

A. Damasio [5] proposed a neuroanatomy model based on experimental evidence to address the binding problem: the integration of both the sensory and motor components in both recognition (perception) and recall. The properties of objects and events that are perceived through the various sensory interactions rely on geographically separate sensory regions of the brain.

He found no structural evidence to support the intuition that temporal and spatial integration occur at a single site. He maintains that the integration of multiple aspects of reality, external and internal, links together distributed repository of fragmented encoded sensory information; stored in remote and geographically separate locations within sensory and motor regions, and reconstructed by co-activation zones. His proposed a neuroanatomical network that allows for both forward propagation and convergence of parallel streams of sensory data with backward propagation of signals back to the points of origin.

The representations of objects with spatial and temporal associations are stored in separate neural regions called *convergence zones*. The reactivation of recall requires the firing of convergence zones with feedback streams propagating from them. Convergence zones *bind* neural activity patterns corresponding to topographically organized fragments. The geographic location of convergence zones varies among individuals but is *not random*.

Figure 9 is a facsimile of a diagram appearing in [6]. The neural architecture depicts the integration of visual (V), somatosensory (SS), and the auditory (A) sensor regions of

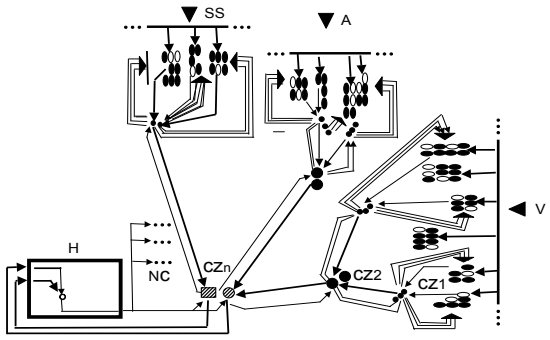

Fig. 9. neuroanatomical network

the cortex. The filled and unfilled dots represent separate functional regions in each of the sensory areas. The arrows pointing from the sensory regions to the convergence zones (CZ1, CZ2, CZn) represent feed-forward paths. Feedback paths from each CZ is represented by arrows pointing back to the sensory regions. The hippocampal system is depicted by H. The outputs of H are returned to CZn and to the non-cortical neural locations of the basal forebrain, brain stem, and the neurotransmitter nuclei. The feed-forward and feedback pathways terminate within the sensory regions over the aggregate of neurons in a distributed fashion, rather than on a specific neuron.

V. Conclusion

The small-world phenomenon has far reaching implications in real-world situations. Maintaining the average distance between vertices to be logarithmic and providing adequate clustering among related nodes are properties that require empirical insight. The small-world models reviewed in this paper relies to some degree upon random graph techniques but in a cleverly constrained fashion. These constraints represent rules that allowed a small range of random flexibility. In many ways, it is surprising how rich the various models are in diversity.

Our primary concern with current small-world models is that the abstraction does not readily address physical constraints or properties that are not necessary evident from the randomness induced in the models. Of course when the models fail, new and better insight usually prevails, so in that sense, small-world models can be a valuable tool.

This paper described the design considerations in the implementation of a neuronal-axon network simulator. The two key features that guided the architectural organization of the final design are the requirements that 1) the growth in the number of edges (communication channels) increases linearly as a function of the number of processors P, and that 2) the architecture must be configured to utilize memory capacity efficiently. It should be pointed out that the emphasis on employing all processors as parallel neuron-engines was never considered. The approach taken

in this work assumed that the best execution time is bound by the slowest executing component of the parallel system. The final network design is organized as a tree network with connections that grow linearly with a characteristic path length $L = O(log_2(P))$. In addition, shortcut edges are introduced to facilitate the high-level routing of activation tokens to remote processors in the tree. The network is reminiscent of Kleinberg's model where long distance shortcuts are avoided. As in the Kleinberg's model, connections are added (not rewired). This also increases the clustering coefficient for selected nodes in the network. A binary tree is used as the base graph but this was for convenience. Quad-trees could also be used with the added property of increasing the clustering coefficient. Unlike Kleinberg's approach, the present design resorted to the application of three-dimensional extension to the original two-dimensional tree network to avoid long distance shortcuts.

Finally, it is argued that physiological evidence supports the organization of neuronal systems as a network of clustered trees.

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