What will borders look like on what condition?

Now that we restrict our universe in two-dimensional space, we use a notation (x, y) instead of (x_1, x_2). So we now express $\mathbf{x} = (x, y)$ Furthermore, both of our two classes are assumed to follow the Gaussian p.d.f. whose μ are $\boldsymbol{\mu}_1 = (0, 0)$ and $\boldsymbol{\mu}_2 = (1, 0)$, and Σ are

$$\Sigma_1 = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}$$

Under this simple condition, our inverse matrix is simply, $|\Sigma_1| = a_1 b_1$ and $|\Sigma_2| = a_2 b_2$. So, we now know

$$\Sigma_1^{-1} = \frac{1}{a_1 b_1} \begin{pmatrix} b_1 & 0 \\ 0 & a_1 \end{pmatrix} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/b_1 \end{pmatrix}$$

and in the same way

$$\Sigma_2^{-1} = \frac{1}{a_2 b_2} \begin{pmatrix} b_2 & 0 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} 1/a_2 & 0 \\ 0 & 1/b_2 \end{pmatrix}$$

Now our Gaussian equation is more specifically

$$p(\mathbf{x}|\omega_1) = \frac{1}{2\pi\sqrt{a_1 b_1}} \exp\{-\frac{1}{2}(x \;\; y) \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/b_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\}$$

and

$$p(\mathbf{x}|\omega_2) = \frac{1}{2\pi\sqrt{a_2 b_2}} \exp\{-\frac{1}{2}(x-1 \;\; y) \begin{pmatrix} 1/a_2 & 0 \\ 0 & 1/b_2 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix}\}.$$

Then we can define our discriminant function $g_i(\mathbf{x})$ $(i = 1, 2)$ taking logarithm based natural number e as

$$g_1(\mathbf{x}) = -\frac{1}{2}(x \;\; y) \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/b_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \ln(2\pi) + \frac{1}{2}\ln(a_1 b_1)$$

and

$$g_2(\mathbf{x}) = -\frac{1}{2}(x-1 \;\; y) \begin{pmatrix} 1/a_2 & 0 \\ 0 & 1/b_2 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} + \ln(2\pi) + \frac{1}{2}\ln(a_2 b_2)$$

Neglecting here the common term for both equation $\ln(2\pi)$, our new discriminant functions are

$$g_1(\mathbf{x}) = -\frac{1}{2}\{\frac{x^2}{a_1} + \frac{y^2}{b_1}\} + \frac{1}{2}\ln(a_1 b_1)$$

and

$$g_2(\mathbf{x}) = -\frac{1}{2}\{\frac{(x-1)^2}{a_2} + \frac{y^2}{b_2}\} + \frac{1}{2}\ln(a_2 b_2)$$

Finally, we obtain the border equation from $g_1(\mathbf{x}) - g_2(\mathbf{x}) = 0$.

$$(\frac{1}{a_1} - \frac{1}{a_2})x^2 + \frac{2}{a_2}x + (\frac{1}{b_1} - \frac{1}{b_2})y^2 = \frac{1}{a_2} + \ln\frac{a_1 b_1}{a_2 b_2} \quad (1)$$

We now know that the shape of the border will be either of the following five cases: (i) straight line (ii) circle; (iii) ellipse; (iV) parabola; (v) hyperbola; (Vi) two straight lines, depending on how the points distribute, that is, depending on a_1, b_1, b_1 and b_2 in our situation above.

Examples

Let's try following calculations,

$$(1) \quad \Sigma_1 = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.10 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.10 \end{pmatrix}$$

$$(2) \quad \Sigma_1 = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.10 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 0.20 & 0 \\ 0 & 0.20 \end{pmatrix}$$

$$(3) \quad \Sigma_1 = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.15 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 0.20 & 0 \\ 0 & 0.25 \end{pmatrix}$$

$$(4) \quad \Sigma_1 = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.15 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 0.15 & 0 \\ 0 & 0.10 \end{pmatrix}$$

$$(5) \quad \Sigma_1 = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.20 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 0.10 & 0 \\ 0 & 0.10 \end{pmatrix}$$

The next example is somewhat tricky. I wanted an example in which the right-hand side of the equation (6) becames zero and the left-hand side is a product of one-order equations of x and y. As you might know, this is the case where border equation will be made up of two straight lines.

$$(6) \quad \Sigma_1 = \begin{pmatrix} 2e & 0 \\ 0 & 0.5 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

My quick calculation tentatively results in as follows. See also the Figure below.

(1) $2x = 1$

(2) $5(x+1)^2 + 5y^2 = 10 - \ln 4$

(3) $5(x+1)^2 + (8/3)y^2 = 10 - \ln(10/3)$

(4) $5(x+1)^2 - (10/3)y^2 = 10$

(5) $20x - 5y^2 = 10 - \ln 2$

(6) $(1 - 1/2e)x^2 - x - y^2 = 0$

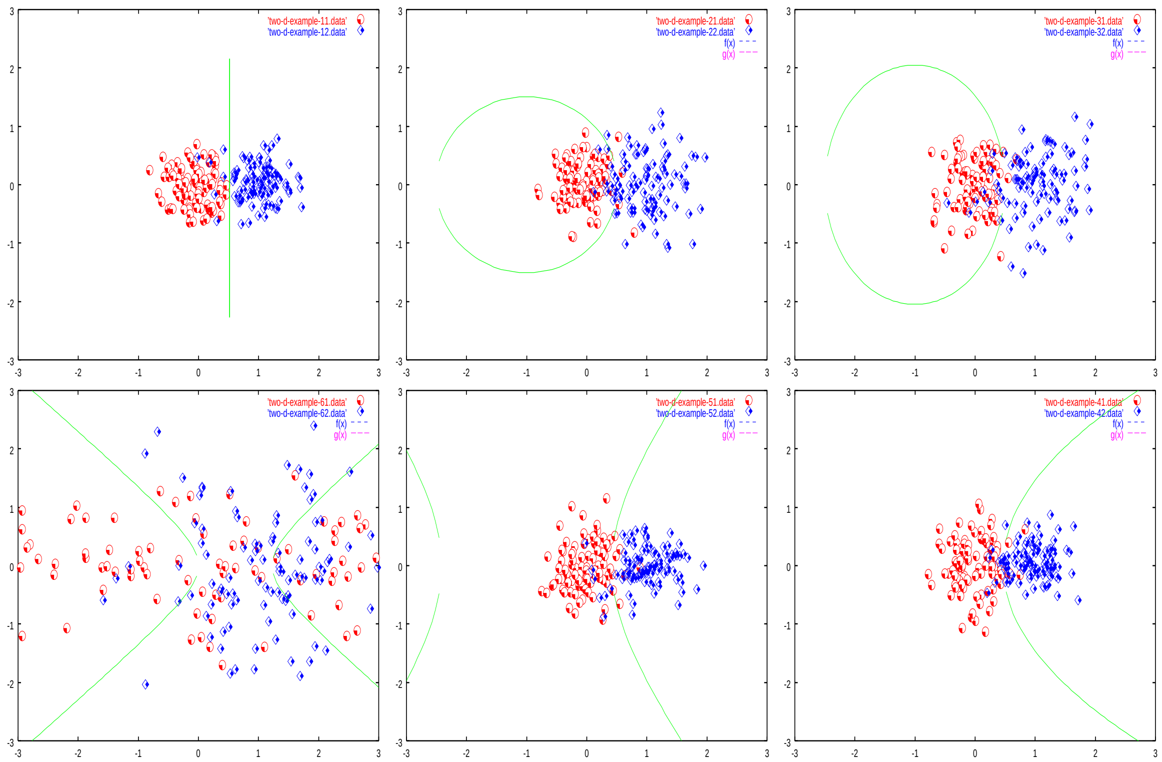

Figure 1: A cloud of 100 points each extracted from a set of two classes and border of the two classes calculated on six different conditions. (Results of (5) and (6) are still fishy and under another trial.)