Fuzzy Logic & Data Processing

Lecture notes for Modern Method of Data Processing (CCOD)
in 2012

Akira Imada
Brest State Technical University, Belarus

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PART I

Fuzzy Set Arithmetics

1 Fuzzy Set Theory

1.1 Fuzzy set vs. Crisp set

- Examples of crisp set
  - ⋆ $0 < x < 10$
  - ⋆ x=12
- Examples of fuzzy set
  - ⋆ $\{x$ is much smaller than 10$\}$
  - ⋆ {x is close to 12}
  - ⋆ Beer is either of {very-cold, cold, not-so-cold, warm}

1.1.1 Membership function

How x is likely to be A is expressed by a function called *membership function.* Usually it is described as $\mu_A(x)$.

For example, a possible membership function for a fuzzy expression {x is close to 12} will be

$$\mu(x) = \frac{1}{1 + (x-12)^2} \tag{1}$$

See Figure 1.

1.1.2 AND and OR in Fuzzy Logic

In the logic of crisp set *A and B* and *A or B* are defined as in Figure 3.
In Fuzzy Logic, on the other hand, the membership function of *A and B* and *A or B* are specified in various way, but most popular ones are:

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \tag{2}$$

and

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \tag{3}$$

respectively.

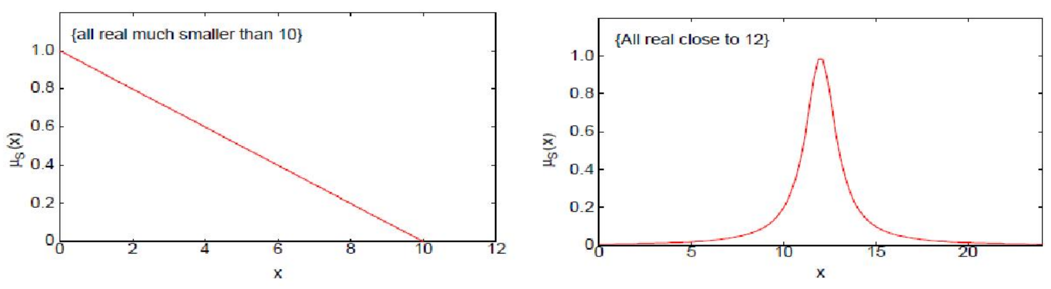


Figure 1: Examples of membership function $\{x$ is much smaller than $10\}$ (right) and {x is close to 12} (left).

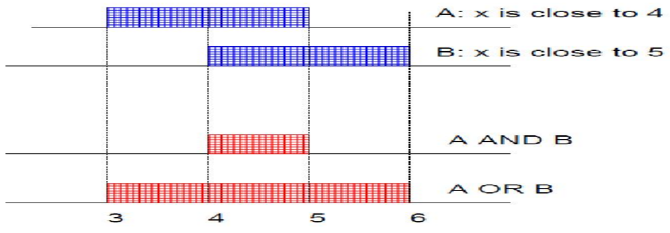


Figure 2: AND and OR in crisp set.

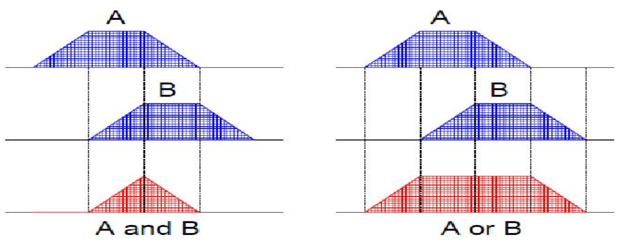


Figure 3: AND and OR in fuzzy set.

To be more concrete the membership function of *x is closer to 4 AND/OR x is closer to 5* is like a Figure 4.

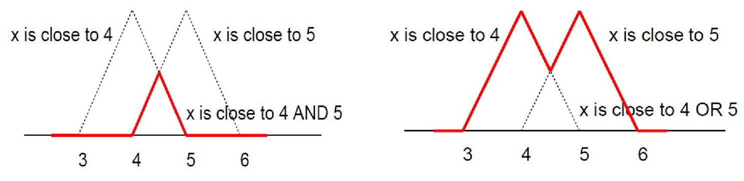
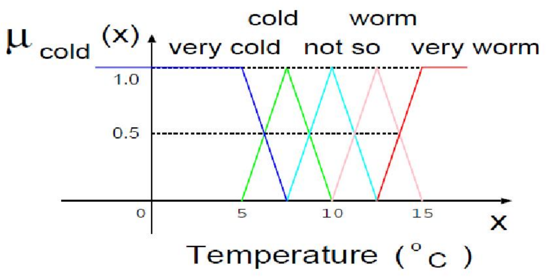


Figure 4: Membership function of *x is closer to 4 OR x is closer to 5*

• **Very cold or pretty cold beer.** ($\mu(x)$ is defined on temperature).
Assume we like very cold beer or pretty cold beer and now we have a beer the temperature of which is 3 degree. Then how is the beer likely to be our prefered one?



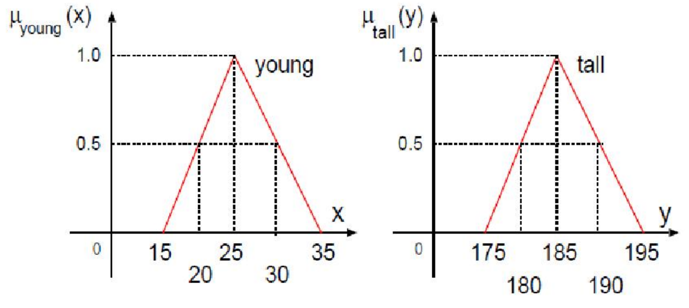
Note that this operation of *OR* was possible because both of the two membership function is defined on the same domain *temperature*. Then what if two membership functions are defined on different domains, such as age and height?

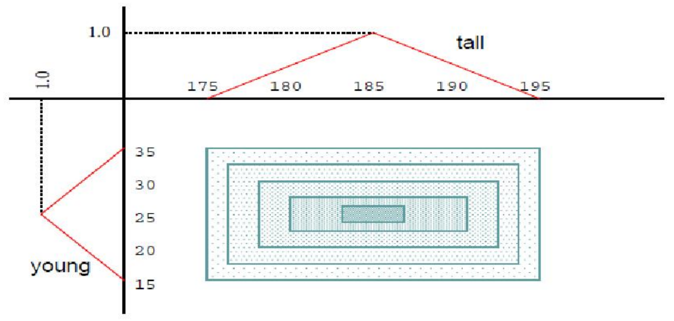
• **Young and tall.**

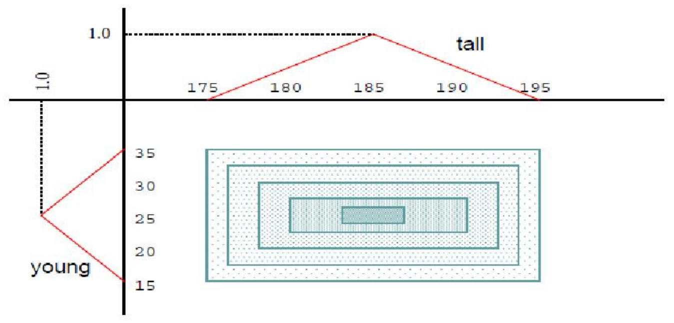
For example,
We cannot draw the membership function of *young and tall* on the 2-dimensional coordinate any more.

(1) 3-D graphic (z = μ is defined on x = age and y = height)

(2) Matrix representation koko


$\mu_{young}(x)$
1.0
0.5
0
young
x
15
20
25
30
35
$\mu_{tall}(y)$
1.0
0.5
0
tall
y
175
180
185
190
195


1.0
tall
175
180
185
190
195
1.0
35
30
25
20
15
young


1.0
tall
175
180
185
190
195
1.0
35
30
25
20
15
young

height \ age	0	10	20	30	40	50	60	70	80	90	100
150											
160											
170											
180											
190											

1.1.3 IF-Then rule in Fuzzy Logic

In Fuzzy Logic, the membership of *IF A Then B* is specified also in many way. Here, let's take it as

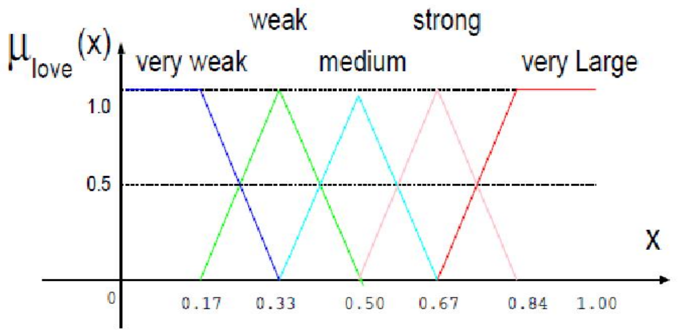
⋆ Mamdani's proposal

$$\mu_{A \to B}(x) = \min\{\mu_A(x), \mu_B(x)\} \tag{4}$$

⋆ Larsen's proposal

$$\mu_{A \to B}(x) = \mu_A(x) \times \mu_B(x) \tag{5}$$

- **If he is young then my love to him is strong.**



- **If he is young and tall then my love to him is very strong.**

1.2 How to express multidimensional membershipfunction

PART II

Fuzzy Controller

2 Fuzzy Controller

Let's construct a virtual metro and control trains by fuzzy controller.

A goal

We now assume x is speed of my car, y is distance to the car in front, and z is how strongly we push brake-pedal. Then let's controll my car with a set of rules, like

- IF x is *high* and y is *short* THEN z should be *strong*
- IF x is *medium* and y is *long* THEN z should be *medium*
- IF x is *medium low* or x is *medium* and y is *long* THEN z should be *weak*
- IF x is *low* or x is *medium low* and y is *short* or y is *medium hort* THEN z should be *medium weak*
- etc.

Then the results will be plotted like in the Figure below.

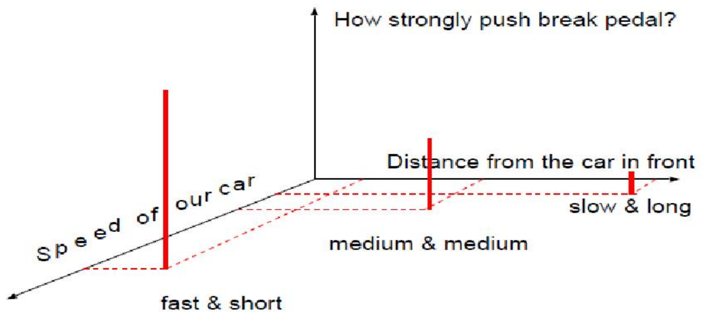
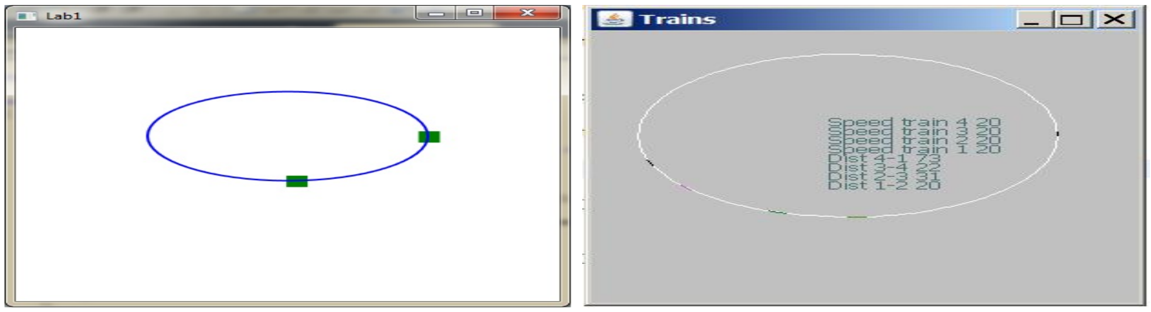

How strongly push break pedal?
Distance from the car in front
Speed of our car
slow & long
medium & medium
fast & short

Figure 5: An example of the goal of Fuzzy controller

2.1 Virtual metro system with two trains in a loop line

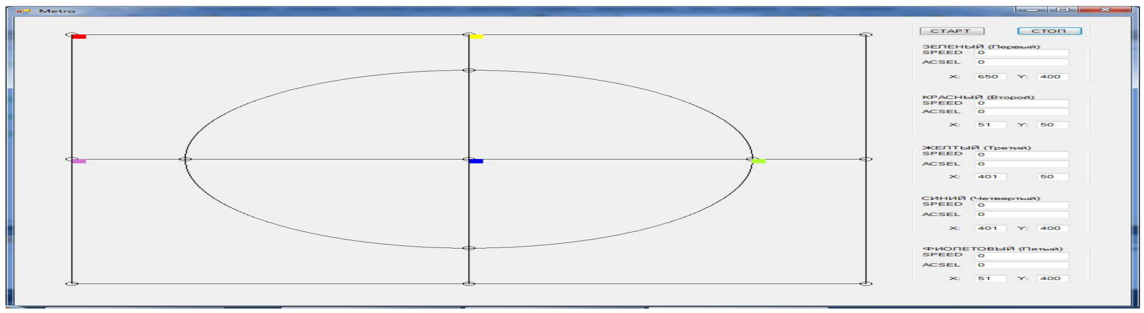
We study Fuzzy Controllar via a simulation of virutal metro with one loop line on which two Train A and B run. To simplify we don't assume stations. That is, both trains always run. The speed of these trains are denoted as x_A and X_B. The distance from train A to train B is denoted as y_A and from train B to train A is y_B. Note that $x_A + y_B$ is constant (length of the loop line). Speed will be controlled by the distance to the train in front via its break. The shorter the distance, the storonger the break in order to avoid a collision.

Exercise 1 *Create your own simulation of metro with one loop on which two trains A and B run, using graphics. 6 parameters x_A, x_B, y_A, y_B, z_A, z_B, should also be desplayed on the screen. The simulation might be started with $x_A = x_B$, $y_A = y_B$, $z_A = z_B = 0$.*



```
Speed train 4 20
Speed train 3 20
Speed train 2 20
Speed train 1 20
Dist 4-1 73
Dist 3-4 22
Dist 2-3 31
Dist 1-2 20
```

Figure 6: An example of a metro line. (Left) a standard implementation of a loop-line with two trains by Moroz Andrey. (Right) Four trains by Dvornichenko Valeriy.



ЗЕЛЕНЫЙ (Первый)
SPEED 0
ACSEL 0
X: 650 Y: 400

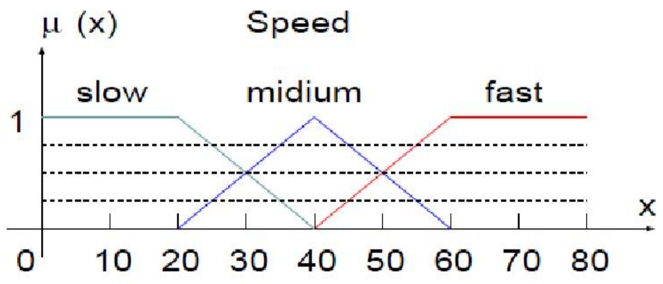
КРАСНЫЙ (Второй)
SPEED 0
ACSEL 0
X: 51 Y: 50

ЖЕЛТЫЙ (Третий)
SPEED 0
ACSEL 0
X: 401 50

СИНИЙ (Четвертый)
SPEED 0
ACSEL 0
X: 401 Y: 400

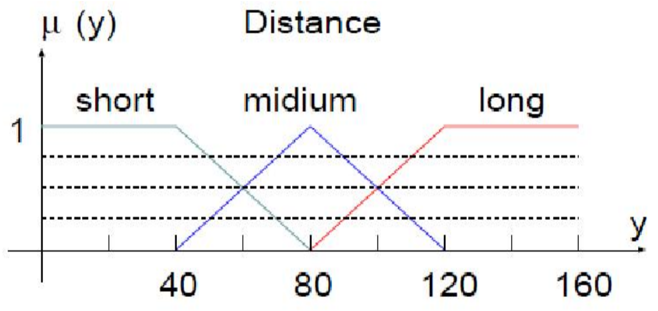
ФИОЛЕТОВЫЙ (Пятый)
SPEED 0
ACSEL 0
X: 51 Y: 400

Figure 7: An example of a more complicated metro lines by Slusareva Maria (2012).



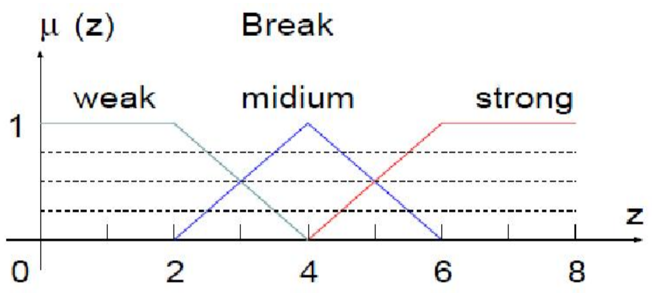
x	0	10	20	30	40	50	60	70	80
μ_{slow}	1.0	1.0	1.0	0.5	0.0	0.0	0.0	0.0	0.0
μ_{medium}	0.0	0.0	0.0	0.5	1.0	0.5	0.0	0.0	0.0
μ_{fast}	0.0	0.0	0.0	0.0	0.5	1.0	1.0	1.0	1.0

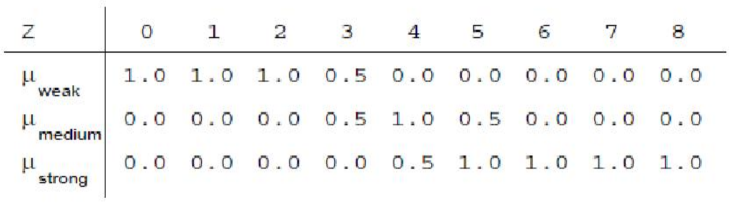
Figure 8: Membership function of x.



y	0	20	40	60	80	100	120	140	160
μ_{short}	1.0	1.0	1.0	0.5	0.0	0.0	0.0	0.0	0.0
μ_{medium}	0.0	0.0	0.0	0.5	1.0	0.5	0.0	0.0	0.0
μ_{long}	0.0	0.0	0.0	0.0	0.5	1.0	1.0	1.0	1.0

Figure 9: Membership function of y.





z	0	1	2	3	4	5	6	7	8
μ_{weak}	1.0	1.0	1.0	0.5	0.0	0.0	0.0	0.0	0.0
μ_{medium}	0.0	0.0	0.0	0.5	1.0	0.5	0.0	0.0	0.0
μ_{strong}	0.0	0.0	0.0	0.0	0.5	1.0	1.0	1.0	1.0

Figure 10: Membership function of z.

2.2 Let's design a set of rules for driving a train

2.2.1 An example of set of 9 rules to control break strength

IF X=FAST AND Y=SHORT THEN Z=STRONG
OR
IF X=FAST AND Y=MEDIUM THEN Z=WEAK
OR
IF X=MEDIUM AND Y=SHORT THEN Z=MEDIUM
OR
IF X=MEDIUM AND Y=MEDIUM THEN Z=STRONG
OR
IF X=SLOW AND Y=SHORT THEN Z=STRONG
...
etc.

2.3 How to construct a membership function of each rule

2.4 Membership function of Z under all possible pair of (X and Y)

Let's assume our rules are

IF x=MEDIUM AND y=MEDIUM then z=MEDIUM
OR

IF x=FAST AND y=SHORT then z=STRONG

Then the membership function of these two rules is

$$\max\{\min\{\mu_{medium}(x)\cdot\mu_{medium}(y),\mu_{medium}(z)\},min\{\mu_{fast}(x)\cdot\mu_{short}(y),\mu_{strong}(z)\}\}$$

For example, let's calculate when $x = 45$ and $y = 70$

z	1	2	3	4	5	6	7	8	9
$\mu(z)$	0.0000	0.0000	0.5000	0.5625	0.5000	0.5625	0.5625	0.5625	0.5625

Thus if every $\mu(z)$ is calculated for all the possinble combination of x and y, we can draw a 3D plot such as

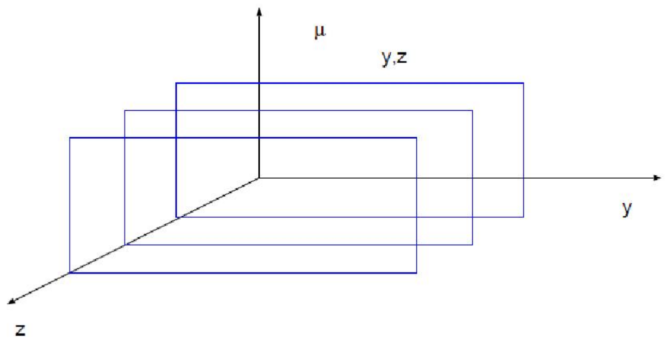


Figure 11: An example of the goal of Fuzzy controller

2.5 Defuzzification

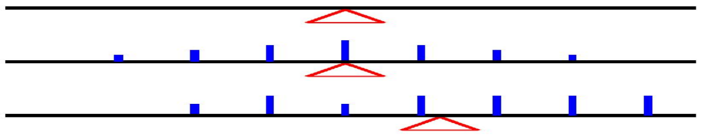


Figure 12: An example of the goal of Fuzzy controller

PART III

Fuzzy Data Mining

3 What is data mining

4 Classify data by a rule set

Assume we classiy M data to be classified by using N features.

$$x_1, x_2, x_3, \cdots, x_N.$$

A rule such as

$$\text{IF} x_1 = A_1 \text{ AND } x_2 = A_2, \text{ AND } \cdots, \text{ AND } x_N = A_N \text{ THEN class is } \omega_p.$$

classifies the data to one class ω_p.

A_i is called attribute. For instance, (i) IF $x_i = 30$, (ii) IF $15 < x_i < 20$, (iii) IF x_i is Large, or (iv) IF x_i is Female, etc. The first two are called *crisp*, second is *fuzzy*, and fourth is called *categorical.* Let's take an example.

$$\text{IF} x_1 = 20g \text{ AND } 10cm < x_2 < 20cm \text{ AND } x_3 = Green, \text{ AND } x_4 = Fruits \text{ THEN this is } apple.$$

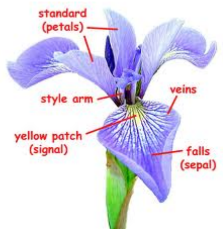
5 A benchmark – Iris database

As an example target here, we classify Iris flowers. Iris flower dataset[1] is made up of 150 samples consists of three species of iris flower, that is, *setosa, versicolor* and *virginica.* Each of these three families includes 50 samples. Each sample is a four-dimensional vector representing four attributes of the iris flower, that is, *sepal-length, sepal-width, petal-length*, and *petal-width.* All data are given as crisp as below.

Our mission is to train our classifiers with known data shown in Appendix as *"data for training"* and then evaluate how good is the classifieyrs with *"data for checking"* also shown in appendix.

Evaluation is how the system appropriately classifies 69 data. Ask system which family by giving 69 data one by one. Score is incremented if the result is correct. Hence the

[1]University of California Urvine Machine Learning Repository.
ics.uci.edu: pub/machine-learning-databases.



x_1	x_2	x_3	x_4	class
0.65	0.80	0.20	0.08	1 (Setosa)
0.62	0.68	0.20	0.08	1 (Setosa)
0.59	0.73	0.19	0.08	1 (Setosa)
0.89	0.73	0.68	0.56	2 (Versicolor)
0.81	0.73	0.65	0.60	2 (Versicolor)
0.87	0.70	0.71	0.60	2 (Versicolor)
0.80	0.75	0.87	1.00	3 (Virginica)
0.73	0.61	0.74	0.76	3 (Virginica)
0.90	0.68	0.86	0.84	3 (Virginica)

maximum score is 69 and minimum is 0. Please note that even random guessing would score one out of three. So score 23 might be the most stupid classifier.

What you should show me are (1) run the algorithm and result of input x1, x2, x3, and x4 (2) a rule set, and (3) success rate (true-positive, true-negative, false-positive, false-negative).

For example

IF$0.81 \leq x_1 \leq 0.89$ AND $0.70 \leq x_1 \leq 0.73$ AND $0.68 \leq x_1 \leq 0.71$ AND $0.56 \leq x_1 \leq 0.60x$

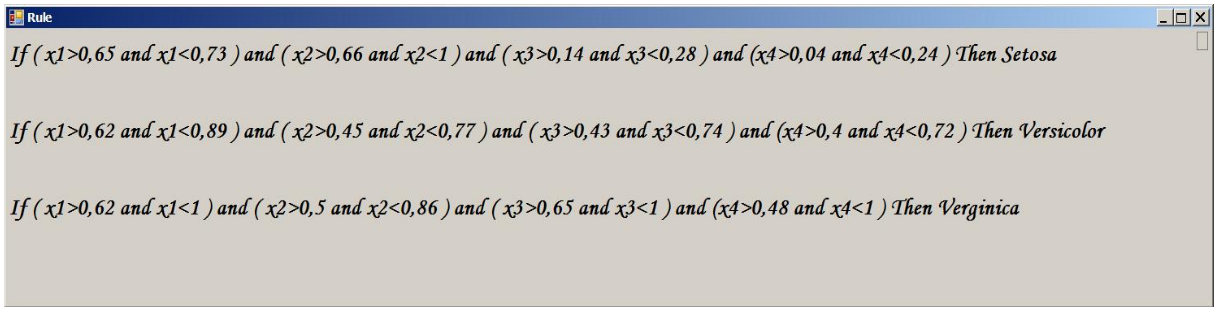
$$\text{THEN this is } Versicolor$$

classifies $x_1 = 0.89$, $x_2 = 0.73$, $x_3 = 0.68$, $x_4 = 0.56$ properly to Versicolor while $x_1 = 0.65$, $x_2 = 0.80$, $x_3 = 0.20$, $x_4 = 0.08$ and $x_1 = 0.80$, $x_2 = 0.75$, $x_3 = 0.87$, $x_4 = 1.00$ are not.

So far so good, but what if the region overlaps with each other between more than two spieces? Or, what if unknown somewhat irregular data are given?

5.0.1 Evaluation of how good is a rule

The rule for class 1 should accept all data of class 1, but at the same time this rule should reject all data of class 2 and class 3. So count (i) how many data from class 1 are

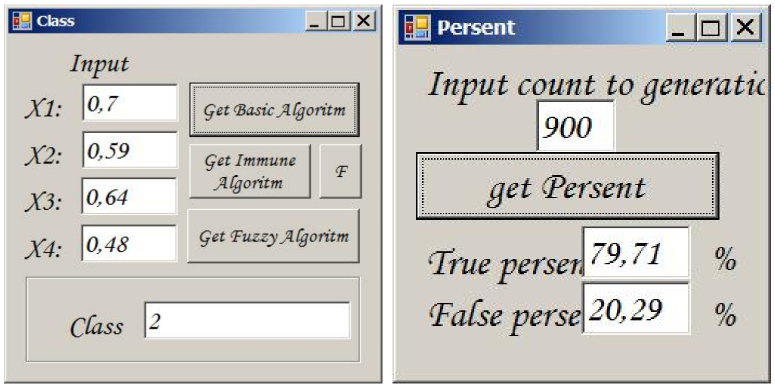


If (x1>0,65 and x1<0,73) and (x2>0,66 and x2<1) and (x3>0,14 and x3<0,28) and (x4>0,04 and x4<0,24) Then Setosa

If (x1>0,62 and x1<0,89) and (x2>0,45 and x2<0,77) and (x3>0,43 and x3<0,74) and (x4>0,4 and x4<0,72) Then Versicolor

If (x1>0,62 and x1<1) and (x2>0,5 and x2<0,86) and (x3>0,65 and x3<1) and (x4>0,48 and x4<1) Then Verginica

Figure 13: An example of a set of the simplest four rules. By Slusareva Maria (2012).



X1: 0,7
X2: 0,59
X3: 0,64
X4: 0,48
Class 2

Input count to generatic
900
True persen 79,71 %
False perse 20,29 %

Figure 14: Example of applying a test data to the rule and its statistics. By Slusareva Maria (2012).

successfully accepted, and then (ii) how many data from class 2 are successfully rejected and (iii) how many data from class 2 are successfully rejected.

5.1 By Genetic Algorithm

Let's the rule is like

$$\text{IF } a_1 < x_1 < a_1 + \delta_1 \text{ AND } a_2 < x_2 < a_2 + \delta_2 \text{ AND } a_3 < x_3 < a_3 + \delta_3 \text{ AND } \\ a_4 < x_4 < a_4 + \delta_4 \text{ THEN class is } p.$$

Then our chromosome will be like

$$(a_1\ \delta_1\ a_2\ \delta_2\ a_3\ \delta_3\ a_4\ \delta_4)$$

where if $a_i + \delta_i > 1$ then we replace the value with 1. For example,

$$(0.23\ 0.41\ 0.52\ 0.81\ 0.89\ 1.00\ 0.11\ 0.72)$$

Create 40 such a chromosome at random which constract a population of the first generation. Then evolve the population with uniform chrossover and mutation under trancate selection with the target being the family 1. Then repeat this with family 2 and 3.

Probably one rule for one family is not enough. Therefore a Multi Objective Optimization will be necessary.

Algorithm 1

```
(1) For p=1 to 4
(2) generate a population of 40 chromosomes at random
(3) select two parents $p_1$ and $p_2$ at random.
(4) create child c by uniform crossover and mutation
(5) If distance(c, $p_1$) $<$ distance(c, $p_2$) and fitness(c¿fitness($p_1$)
      Then replace $p_1$ with c
(6) ElseIf distance(c, $p_1$) $>$ distance(c, $p_2$) and fitness(c¿fitness($p_2$)
      Then replace $p_2$ with c
(7) Repeat (3)-(6) until all the chromosom is replaced
(8) Repeat (3)-(7) until all the chromosom never be changed
(9) End For
```

5.1.1 Fitness

Fitness might be evaluated by

$$f = \frac{TP}{TP + FN} \times \frac{TN}{FP + TN} \tag{6}$$

where TP stands for true-positive, FN stands for false-negative, TN stands for true-negative, and FP stands for false-positive, and true-positive is the number of cases covered by the rule that have the class predicted by the rule; false-positives is the number of cases covered by the rule that have a class different from the class predicted by the rule; false-negatives is the number of cases that are not covered by the rule but that have the class predicted by the rule; true-negatives is the number of cases that are not covered by the rule and do not have the class predicted by the rule.

5.1.2 Result to be shown

	Setosa		Versicolor		Virginica	
	yes	no	yes	no	yes	no
rule-01 for class 1	100	0	0	100	0	100
rule-02 for class 1	100	0	0	100	0	100
rule-03 for class 1	100	0	0	100	0	100
rule-04 for class 1	100	0	0	100	0	100
rule-05 for class 2	0	100	100	0	0	100
rule-06 for class 2	0	100	100	0	0	100
rule-07 for class 2	0	100	100	0	0	100
rule-08 for class 2	0	100	100	0	0	100
rule-09 for class 3	0	100	0	100	100	0
rule-10 for class 3	0	100	0	100	100	0
rule-11 for class 3	0	100	0	100	100	0
rule-12 for class 3	0	100	0	100	100	0

PART III

Fuzzy Neural Network with Takagi-Sugeno Model

5.2 A Fuzzy Neural Network Approach

The goal is to classify the data taken from the n-dimensional data-set into either of the pre-defined m classes. For the purpose, Castellano et al. [?] used the inference mechanism of the zero-order Takagi-Sugeno fuzzy model; then realized the idea by a fuzzy neural network model. To train the fuzzy neuronal network, they employed a combination of (i) *a competitive learning* to determine the architecture of the fuzzy neural network at first and (ii) *a gradient descent learning* to optimize the synaptic weights afterwards. We, on the other hand, employ an evolutionary computation technique to train the network, since we already know the optimal network structure under our current interest, and as such, our concern is just to obtain the solution of weight configuration of the fuzzy neural network.

In the following three sub-subsections, Takagi-Sugeno fuzzy model, a realization of the model by fuzzy neural network, and how we optimize the weight of the fuzzy neural network by an evolutionary computation are described more in detail.

5.2.1 Takagi-Sugeno Model.

Though Castellano et al. [?] stated the method very clearly in their paper, let us briefly describe it with an intention of making this paper self-contained. Takagi-Sugeno fuzzy inference model is made up of a set of H rules, such as

$$R_k\text{: IF } (x_1 \text{ is } A_1^k) \text{ and } \cdots \text{ and } (x_n \text{ is } A_n^k)$$

$$\text{THEN } (y_1 \text{ is } \nu_{k1}) \text{ and } \cdots \text{ and } (y_m \text{ is } \nu_{km})$$

where R_k is the k-th rule ($k = 1, \cdots H$), x_i denotes the i-th variable of the input data ($i = 1, \cdots, n$), y_j denotes the j-th output variable ($j = 1, \cdots, m$), A_i^k denotes a fuzzy set which is usually expressed by a linguistic term such as *"Medium-Large"* but here expressed by a shape of membership function defined one by one on the corresponding input variable, and ν_{kj} denotes a fuzzy singleton each defined on the output variables indicating the likeliness of how the input belongs to the j-th class according to the k-th rule.

A_i^k is defined by Gaussian membership functions

$$\mu_{ik}(x_i) = \exp\{-(x_i - w_{ik})^2/\sigma_{ik}^2\}.$$

Then defuzzification for an input $\mathbf{x}^0 = (x_1^0, \cdots, x_n^0)$ is via the equation:

$$y_j^0 = \{\sum_{k=1}^{H} (\mu_k(\mathbf{x}^0) \cdot \nu_{kj})\} / \sum_{k=1}^{H} \mu_k(\mathbf{x}^0)$$

where

$$\mu_k(\mathbf{x}^0) = \prod_{i=1}^{n} \mu_{ik}(x_i^0)$$

is the results of application of the Larsen product operator.

In other words, the procedure of inference is as follows. When an input $\mathbf{x} = (x_1, \cdots x_n)$ is given, each of the H rules evaluates the $\mathbf{x}$ and output the likeliness of the class, from one class to the next, to which $\mathbf{x}$ belongs to. The evaluation by k-th rule of x_i is by the corresponding membership function $\mu_{ik}(x_i)$ which is specified by giving two parameters w_{ik} and σ_{ik} so that it returns a value ranging from 0 to 1. See, e.g., Fig. 1 where the i-th coordinate of the input $\mathbf{x}$ is evaluated by A_i^k, the i-th antecedent of the *IF* part of the Rule$_k$, which is represented by a membership function not by a usual linguistic term like *"Small"*. The returned membership value in this example in the figure is 0.71, suggesting, say, "The likeliness of if the variable is "Medium Large" is 0.71."

Figure 15: A fictitious sketch of an evaluation of x_i, the i-th entry of the input $\mathbf{x}$, by the i-th antecedent part of the k-th rule A_i^k.

Using those n values of $\mu_{ik}(x_i)$, each of the H rules calculates $\mu_k(\mathbf{x})$, and finally these H values are combined to calculate m values of y_j, the resultant defuzzified value for each of the m classes.

5.2.2 Fuzzy Neural Network Implementation.

The procedure described in the previous sub-subsection can be realized when we assume a neural network architecture such as depicted in Fig. 2. The 1st layer is made up of n input neurons. The 2nd layer is made up of H groups of a neuronal structure each contains n neurons where the i-th neuron of the k-th group has a connection to the i-th neuron in the 1st layer with a synaptic connection which has a pair of weights (w_{ik}, σ_{ik}). Then k-th group in the second layer calculates the value $\mu_k(\mathbf{x})$ from the values which are received from each of the n neurons in the first layer. The 3rd layer is made up of m neurons each of which collects the H values from the output of the second layer, that is

j-th neuron of the 3rd layer receives the value from k-th output in the second layer with the synapse which has the weight ν_{kj}

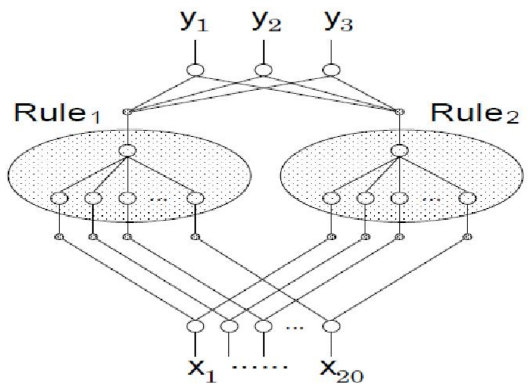


Figure 16: Architecture of the proposed fuzzy neural network which infers how an input $\mathbf{x} = (x_1, \cdots x_n)$ is likely to belong to the j-th class by generating outputs y_j each of which reflect the degree of the likeliness. In this example, a 20-dimension data input will be inferred to which of the 3 classes the input belongs by using 2 rules.

5.2.3 How it learns?

Castellano et al. [?] used (i) a competitive learning to determine how many rules are needed under initial weights created at random. Then, in order to optimize the initial random weight configuration, they use (ii) a gradient method performing the steepest descent on a surface in the weight space employing the same training data, that is, supervised learning.

Here, on the other hand, we use a simple genetic algorithm, since our target space is specific enough to know the network structure in advance, i.e., only unique rule is necessary. Our concern, therefore, is just obtaining the solution of weight configuration of the network. That is to say, all we want to know is a set of parameters w_{ik}, σ_{ik} and ν_{kj} $(i = 1, \cdots n)$, $(k = 1, \cdots H)$, $(j = 1, \cdots m)$ where n is the dimension of data, H is the number of rules, and m is the number of outputs. Hence our chromosome has those $n \times H \times m$ genes. Starting with a population of chromosomes whose genes are randomly created, they evolve under *simple truncate selection* where higher fitness chromosome are chosen, with *uniform crossover* and occasional *mutation* by replacing some of a few genes with randomly created other parameters, expecting higher fitness chromosomes will be emerged. These settings are determined by trials and errors experimentally.

PART IV

Fuzzy Relation

6 Fuzzy Relation

In this section we study fuzzy expressions such as "at least middle-aged," brighter than average," more or less expensive" and "younger than about 20."

First, let's recall Cartesian product $X \times Y$ in which both X and Y is a set. Let me take an example. Assume now $X = \{1, 2\}$ and $Y = \{a, b, c\}$ then $X \times Y = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$. Then relation is defiened over Cartesian product $X \times Y$, that is, a subset of $X \times Y$. In other words relation is a set of ordered pair in which order is important.

Generally it is defined over multipel set, like $X_1 \times X_2 \times \cdots \times X_n$, but here we think of only product of two set, and call it *binary relation.*

To visualize we can plot $\mu_R(X, Y)$ 3-D Cartesian space.

⋆ Example 1 ... $X = \{1, 2\}$, $Y = \{2, 3, 4\}$, $R : X < Y$

Let's think of it as a *crisp* logic, that is, the value is 1 (yes) or 0 (no). Then membership function of this relation will be:

$X \setminus Y$	2	3	4
1	1	0	0
2	1	1	0
3	0	0	1

Then what about the relation $R : x \approx y$. Let's think of this example with *fuzzy* logic.

⋆ Example 2 ... $X = \{1, 2\}$, $Y = \{2, 3, 4\}$, $R : X \approx Y$

$X \setminus Y$	2	3	4
1	2/3	1/3	0
2	1	2/3	1/3
3	2/3	1	2/3

The values are just examples. Further more we think of X and Y as a continuous values instead of integer. Then membership function is a surface instead of just 9 points, over $X - Y$ coordinate.

We now proceed to examples where we use fuzzy linguistic expression instead of numbers.

$\star$ Example 3 ... X = {green, yellow, red}, Y = {unripe, semiripe, ripe}.

Imagine an apple. First, with a *crisp* logic. A red apple is usually ripe but a green apple is unripe. Thus:

$X \setminus Y$	unripe	semiripe	ripe
green	1	0	0
yellow	0	1	0
red	0	0	1

Now, secondly, with a *fuzzy* logic. A red apple is *provably* ripe, but a green apple is *most likely*, and so on. Thus, for example:

$X \setminus Y$	unripe	semiripe	ripe
green	1	0.5	0
yellow	0.3	1	0.4
red	0	0.2	1

These matrices are not necessarily rectangular. For example:

$\star$ Example 4 ... X = {Brest, London, BuenosAires} Y=Tokyo, NewYork, Minsk, Johanesburg R: very far.

$X \setminus Y$	Tokio	New York	Minsk	Johanesburg
Brest				
London				
Buenos Aires				

Try to fill those blancs by yourself.

6.1 Combine two fuzzy relations

We now return to the previous example of tomato.

$X \setminus Y$	unripe	semiripe	ripe
green	1	0.5	0
yellow	0.3	1	0.4
red	0	0.2	1

This is the relation of two sets:

$$X = \{green, yellow, red\}$$

and

$$Y = \{unripe, semiripe, ripe\}$$

Let's call this relation R_1. Then we think a similar but new Relation.

$$Y = \{unripe, semiripe, ripe\}$$

and

$$Z = \{sour, sour - sweet, sweet\}$$

Let's call this relation R_2.

$X \setminus Y$	sour	sour-sweet	sweet
unripen	0.8	0.5	0.1
semiripe	0.1	0.7	0.5
ripe	0.2	0.3	0.9

If we combine these two relations R_1 and R_2 by the formula

$$\mu_R(x, z) \geq \max_{y \in X}\{\min\{\mu_R(x, y), \mu_R(y, z)\}\},$$

the result is:
This relation could be expressed by our daily language like

> "If tomato is red then it's most likely sweet , possibly sour-sweet, and unlikely sour."

$X \setminus Y$	sour	sour-sweet	sweet
red	0.8	0.5	0.5
yellow	0.3	0.7	0.5
green	0.2	0.3	0.9

> "If tomato is yellow then probably it's sour-sweet , possibly sour, maybe sweet."

> "If tomato is green then almost always sour, less likely sour-sweet, unlikely sweet."

Or, we could say:

> "Now tomato is more or less red, then what is taste like?"

Other than

$$\mu_R(x,z) = \max_{y\in X}\{\min\{\mu_R(x,y), \mu_R(y,z)\}\},$$

We have

$$\mu_R(x,z) = \max_{y\in X}\{(mu_R(x,y) + \mu_R(y,z))/2\},$$

$$\mu_R(x,z) = \max_{y\in X}\{\min\{\mu_R(x,y) \times \mu_R(y,z)\}\}.$$

We call the above MAX-MIN, MAX-AVG, MAX-PROD, respectively.

Exercise 2 *Calculate $R_* R_2$ by (i) MAX-MIN (ii) MAX-AVG, and (iii) MAX-PROD.*

$$R_1 = \begin{pmatrix} 0.9 & 0.8 & 0.7 \\ 0.8 & 0.7 & 0.6 \\ 0.7 & 0.6 & 0.5 \end{pmatrix}.$$

$$R_2 = \begin{pmatrix} 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.5 \\ 0.4 & 0.5 & 0.6 \end{pmatrix}.$$

6.2 Classification by Fuzzy Relation

In this section, we consider $X * X$, not $X * Y$. That is, Relation of identical set. As our purpose is classification, relation is how similar between two elements of one set. We have the following definitions of how close the two elements are.

- similarity;
- proximity;
- equivalent; and
- tolerant.

Definition 1 (Similarity Relation) *If the relation R fulfil the following two conditions R is Similarity Relation.*
(i) reflecxivity

$$\mu_R(x,x) = 1 ... for \forall x \in X$$

(ii) symmetry

$$\mu_R(x,y) = \mu_R(y,x) ... for \forall x,y \in X$$

Let me give an example of similarity relation:

$$R_2 = \begin{pmatrix} 1 & & \\ 0.8 & 1 & \\ 0.7 & 0.2 & 1 \end{pmatrix}.$$

Hereafter as matrix is symmetry so only half of the elements as well as diagonal will be shown.

Exercise 3 *Calculate composition of the following relation*

$$R = \begin{pmatrix} 1 & & \\ 0.8 & 1 & \\ 0.7 & 0.2 & 1 \end{pmatrix}.$$

Starting with a similarity relation R^0 we can create a series of relations by composition. Let's denote i-th relation created in this way R^n.

Theorem 1

$$R^0 \leq R^1 \leq R^2 \cdots$$

where

Definition 2 *When all elements of R_1 is smaller than or equal to all corresponding element of R_2 we describe it as:*

$$R_1 \leq R_2$$

Theorem 2 *When we repeat this composition and finally obtained the results $R^{(n)} = R^{(n+1)}$, not $R^{(n)} < R^{(n+1)}$ then $R^(n)$ is tolerant relation.*

Definition 3 *For $0 < a \leq 1$, $R_\alpha\{(x,y)|\mu_R(x,y) \leq \alpha\}$ is called an α cut of relation R.*

Theorem 3 *For any fuzz relation R on $X \times Y$,*

$$R = \sum_\alpha \alpha R_\alpha, 0 < \alpha \leq 1$$

where $\mu_{\alpha R_\alpha(x,y) = \alpha \mu_{R_\alpha}(x,y} = \alpha$ if $(x,y) \in R_\alpha$, and o otherwise.

Theorem 4 *if R is a MAX-MIN similarity then for any $0 < \alpha \leq 1$ then R_α should be an equivalence relation.*

Definition 4 *A crisp relation R on X is called tolerance relation if*

$$\mu_R(x,x) = 1$$

and

$$\mu_R(x,y) \Rightarrow \mu_R(y,x) = 1$$

Theorem 5 *If R is a proximity relation then for any $0 < \alpha \leq 1$ then R_α is a tolernace relation.*

Example 1 *Now a proximity relation $R^{(0)}$ on $X = \{x_1, x_2, x_3\}$ is*

$$R^{(0)} = \begin{pmatrix} 1 & & \\ 0.8 & 1 & \\ 0.7 & 0.2 & 1 \end{pmatrix}.$$

Then

$$R^{(1)} = \begin{pmatrix} 1 & & \\ 0.8 & 1 & \\ 0.7 & 0.7 & 1 \end{pmatrix} = R^{(2)}.$$

is a MAX-MIN similarity relation on MAX-MIN composition. And

$$R^{(1)} = \alpha_1 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \bigcup \alpha_2 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bigcup \alpha_3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $0 < \alpha_1 \leq 0.7$*,* $0.7 < \alpha_2 \leq 0.8$ *and* $0.8 < \alpha_3 \leq 1$*.*
Then $R^{(1)}$ *classifies as follows:* $0 < \alpha \leq 0.7 \Rightarrow \{x_1, x_2, x_3\}$,
$0.7 < \alpha \leq 0.8 \Rightarrow \{x_1, x_2\}, \{x_3\}$,
$0.8 < \alpha \leq 1 \Rightarrow \{x_1\}, \{x_2\}, \{x_3\}$.

Exercise 4 *Apply the above to:*

$$R = \begin{pmatrix} 1 & & & & & & & & & \\ 0.2 & 1 & & & & & & & & \\ 0.5 & 0.3 & 1 & & & & & & & \\ 0.8 & 0.6 & 0.5 & 1 & & & & & & \\ 0.6 & 0.7 & 0.3 & 0.7 & 1 & & & & & \\ 0.2 & 0.9 & 0.4 & 0.3 & 0.2 & 1 & & & & \\ 0.3 & 0.2 & 0.1 & 0.5 & 0.4 & 0.1 & 1 & & & \\ 0.9 & 0.8 & 0.3 & 0.4 & 0.5 & 0.3 & 0.6 & 1 & & \\ 0.4 & 0.3 & 0.7 & 0.1 & 0.8 & 0.7 & 0.1 & 0 & 1 & \\ 0.3 & 0.2 & 0.6 & 0.3 & 0.9 & 0.2 & 0.3 & 0.2 & 0.1 & 1 \end{pmatrix}.$$

APPENDIX

Iris Flower Database

Data for training - 3×40 = 120 data

Setosa				Versicolor				Virginica			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.65	0.80	0.20	0.08	0.89	0.73	0.68	0.56	0.80	0.75	0.87	1.00
0.62	0.68	0.20	0.08	0.81	0.73	0.65	0.60	0.73	0.61	0.74	0.76
0.59	0.73	0.19	0.08	0.87	0.70	0.71	0.60	0.90	0.68	0.86	0.84
0.58	0.70	0.22	0.08	0.70	0.52	0.58	0.52	0.80	0.66	0.81	0.72
0.63	0.82	0.20	0.08	0.82	0.64	0.67	0.60	0.82	0.68	0.84	0.88
0.68	0.89	0.25	0.16	0.72	0.64	0.65	0.52	0.96	0.68	0.96	0.84
0.58	0.77	0.20	0.12	0.80	0.75	0.68	0.64	0.62	0.57	0.65	0.68
0.63	0.77	0.22	0.08	0.62	0.55	0.48	0.40	0.92	0.66	0.91	0.72
0.56	0.66	0.20	0.08	0.84	0.66	0.67	0.52	0.85	0.57	0.84	0.72
0.62	0.70	0.22	0.04	0.66	0.61	0.57	0.56	0.91	0.82	0.88	1.00
0.68	0.84	0.22	0.08	0.63	0.45	0.51	0.40	0.82	0.73	0.74	0.80
0.61	0.77	0.23	0.08	0.75	0.68	0.61	0.60	0.81	0.61	0.77	0.76
0.61	0.68	0.20	0.04	0.76	0.50	0.58	0.40	0.86	0.68	0.80	0.84
0.54	0.68	0.16	0.04	0.77	0.66	0.68	0.56	0.72	0.57	0.72	0.80
0.73	0.91	0.17	0.08	0.71	0.66	0.52	0.52	0.73	0.64	0.74	0.96
0.72	1.00	0.22	0.16	0.85	0.70	0.64	0.56	0.81	0.73	0.77	0.92
0.68	0.89	0.19	0.16	0.71	0.68	0.65	0.60	0.82	0.68	0.80	0.72
0.65	0.80	0.20	0.12	0.73	0.61	0.59	0.40	0.97	0.86	0.97	0.88
0.72	0.86	0.25	0.12	0.78	0.50	0.65	0.60	0.97	0.59	1.00	0.92
0.65	0.86	0.22	0.12	0.71	0.57	0.57	0.44	0.76	0.50	0.72	0.60
0.68	0.77	0.25	0.08	0.75	0.73	0.70	0.72	0.87	0.73	0.83	0.92

(to be cont'd to the next page)

(cont'd)

Setosa				Versicolor				Virginica			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.65	0.84	0.22	0.16	0.77	0.64	0.58	0.52	0.71	0.64	0.71	0.80
0.58	0.82	0.14	0.08	0.80	0.57	0.71	0.60	0.97	0.64	0.97	0.80
0.65	0.75	0.25	0.20	0.77	0.64	0.68	0.48	0.80	0.61	0.71	0.72
0.61	0.77	0.28	0.08	0.81	0.66	0.62	0.52	0.85	0.75	0.83	0.84
0.63	0.68	0.23	0.08	0.84	0.68	0.64	0.56	0.91	0.73	0.87	0.72
0.63	0.77	0.23	0.16	0.86	0.64	0.70	0.56	0.78	0.64	0.70	0.72
0.66	0.80	0.22	0.08	0.85	0.68	0.72	0.68	0.77	0.68	0.71	0.72
0.66	0.77	0.20	0.08	0.76	0.66	0.65	0.60	0.81	0.64	0.81	0.84
0.59	0.73	0.23	0.08	0.72	0.59	0.51	0.40	0.91	0.68	0.84	0.64
0.61	0.70	0.23	0.08	0.70	0.55	0.55	0.44	0.94	0.64	0.88	0.76
0.68	0.77	0.22	0.16	0.70	0.55	0.54	0.40	1.00	0.86	0.93	0.80
0.66	0.93	0.22	0.04	0.73	0.61	0.57	0.48	0.81	0.64	0.81	0.88
0.70	0.95	0.20	0.08	0.76	0.61	0.74	0.64	0.80	0.64	0.74	0.60
0.62	0.70	0.22	0.04	0.68	0.68	0.65	0.60	0.77	0.59	0.81	0.56
0.63	0.73	0.17	0.08	0.76	0.77	0.65	0.64	0.97	0.68	0.88	0.92
0.70	0.80	0.19	0.08	0.85	0.70	0.68	0.60	0.80	0.77	0.81	0.96
0.62	0.70	0.22	0.04	0.80	0.52	0.64	0.52	0.81	0.70	0.80	0.72
0.56	0.68	0.19	0.08	0.71	0.68	0.59	0.52	0.76	0.68	0.70	0.72
0.65	0.77	0.22	0.08	0.70	0.57	0.58	0.52	0.87	0.70	0.78	0.84

Data for evaluating the system after training - $3 \times 23 = 69$ data

Setosa				Versicolor				Virginica			
x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
0.63	0.80	0.19	0.12	0.70	0.59	0.64	0.48	0.85	0.70	0.81	0.96
0.57	0.52	0.19	0.12	0.77	0.68	0.67	0.56	0.87	0.70	0.74	0.92
0.56	0.73	0.19	0.08	0.73	0.59	0.58	0.48	0.73	0.61	0.74	0.76
0.63	0.80	0.23	0.24	0.63	0.52	0.48	0.40	0.86	0.73	0.86	0.92
0.65	0.86	0.28	0.16	0.71	0.61	0.61	0.52	0.85	0.75	0.83	1.00
0.61	0.68	0.20	0.12	0.72	0.68	0.61	0.48	0.85	0.68	0.75	0.92
0.65	0.86	0.23	0.08	0.72	0.66	0.61	0.52	0.80	0.57	0.72	0.76
0.58	0.73	0.20	0.08	0.78	0.66	0.62	0.52	0.82	0.68	0.75	0.80
0.67	0.84	0.22	0.08	0.65	0.57	0.43	0.44	0.78	0.77	0.78	0.92
0.63	0.75	0.20	0.08	0.72	0.64	0.59	0.52	0.75	0.68	0.74	0.72
0.62	0.70	0.22	0.04	0.80	0.52	0.64	0.52	0.81	0.70	0.80	0.72
0.56	0.68	0.19	0.08	0.71	0.68	0.59	0.52	0.76	0.68	0.70	0.72
0.65	0.77	0.22	0.08	0.70	0.57	0.58	0.52	0.87	0.70	0.78	0.84
0.63	0.80	0.19	0.12	0.70	0.59	0.64	0.48	0.85	0.70	0.81	0.96
0.57	0.52	0.19	0.12	0.77	0.68	0.67	0.56	0.87	0.70	0.74	0.92
0.56	0.73	0.19	0.08	0.73	0.59	0.58	0.48	0.73	0.61	0.74	0.76
0.63	0.80	0.23	0.24	0.63	0.52	0.48	0.40	0.86	0.73	0.86	0.92
0.65	0.86	0.28	0.16	0.71	0.61	0.61	0.52	0.85	0.75	0.83	1.00
0.61	0.68	0.20	0.12	0.72	0.68	0.61	0.48	0.85	0.68	0.75	0.92
0.65	0.86	0.23	0.08	0.72	0.66	0.61	0.52	0.80	0.57	0.72	0.76
0.58	0.73	0.20	0.08	0.78	0.66	0.62	0.52	0.82	0.68	0.75	0.80
0.67	0.84	0.22	0.08	0.65	0.57	0.43	0.44	0.78	0.77	0.78	0.92
0.63	0.75	0.20	0.08	0.72	0.64	0.59	0.52	0.75	0.68	0.74	0.72